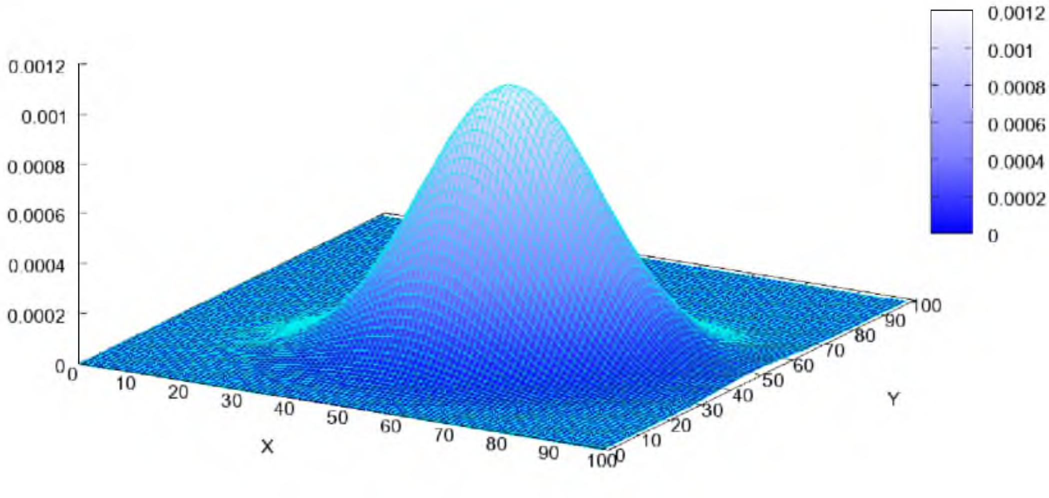
Xashimov. A.R

EHTIMOLLAR NAZARIYASI VA MATEMATIK STATISTIKA



Toshkent-2010

Mundarija

IV bob. Tasodifiy miqdorlarning funksiyalari

V bob. Ehtimollar nazariyasining limit teoremalari

MATEMATIK STATISTIKA

VI bob. Tanlanma va uning xarakteristikalari

VII bob. Noma’lum parametrlarni baholash

VIII bob. Statistik gipotezalarni tekshrish

IX bob. Ko‘p o‘lchovli statistik tahlil usullari

I bob. Tasodifiy hodisalar

1.1 Ehtimollar nazariyasining predmeti

Ehtimollar nazariyasi “tasodifiy tajribalar”, ya’ni natijasini oldindan aytib bo‘lmaydigan tajribalardagi qonuniyatlatni o‘rganuvchi matematik fandir. Bunda shunday tajribalar qaraladiki, ularni o‘zgarmas (ya’ni, bir xil) shartlar kompleksida hech bo‘lmaganda nazariy ravishda ixtiyoriy sonda takrorlash mumkin, deb hisoblanadi. Bunday tajribalar har birining natijasi *tasodifiy hodisa* ro‘y berishidan iboratdir. Insoniyat faoliyatining deyarli hamma sohalarida shunday holatlar mavjudki, u yoki bu tajribalarni bir xil sharoitda ko‘p matra takrorlash mumkin bo‘ladi. Ehtimollar nazariyasini sinovdan-sinovga o‘tishida natijalari turlicha bo‘lgan tajribalar qiziqtiradi. Biror tajribada ro‘y berish yoki bermasligini oldindan aytib bo‘lmaydigan hodisalar tasodifiy hodisalar deyiladi. Masalan, tanga tashlash tajribasida har bir tashlashga ikki tasodifiy hodisa mos keladi: tanganing gerb tomoni tushishi yoki tanganing raqam tomoni tushishi. Albatta, bu tajribani bir marta takrorlashda shu ikki tasodifiy hodisalardan faqat bittasigina ro‘y beradi. Tasodifiy hodisalarni biz tabiatda, jamiatda, ilmiy tajribalarda, sport va qimor o‘yinlarida kuzatishimiz mumkin. Umumlashtirib aytish mumkinki, tasodifiyat elementlarisiz rivojlanishni tasavvur qilish qiyindir. Tasodifiyatsiz umuman hayotning va biologik turlarning yuzaga kelishini, insoniyat tarihini, insonlarning ijodiy faoliyatini, sotsial-iqtisodiy tizimlarning rivojlanishini tasavvur etib bo‘lmaydi. Ehtimollar nazariyasi esa aynan mana shunday tasodifiy bog‘liqliklarning matematik modelini tuzish bilan shug‘illanadi. Tasodifiyat insoniyatni doimo qiziqtirib kelgandir. Shu sababli ehtimollar nazariyasi boshqa matematik fanlar kabi amaliyot talablariga mos ravishda rivojlangan. Ehtimollar nazariyasi boshqa matematik fanlardan farqli o‘laroq nisbatan qisqa, ammo o‘ta shijoatlik rivojlanish tarixiga ega. Endi qisqacha tarixiy ma’lumotlarni keltiramiz. Ommaviy tasodifiy hodisalarga mos masalalarni sistematik ravishda o‘rganish va ularga mos matematik apparatning yuzaga kelishi XVII asrga to‘g‘ri keladi. XVII asr boshida, mashhur fizik Galiley fizik o‘lchashlardagi xatoliklarni tasodifiy deb hisoblab, ularni ilmiy tadqiqot qilishga uringan. Shu davrlarda kasallanish, o‘lish, baxtsiz hodisalar statistikasi va shu kabi ommaviy tasodifiy hodisalardagi qonuniyatlarni tahlil qilishga asoslangan sug‘urtalanishning umumiy nazariyasini yaratishga ham urinishlar bo‘lgan. Ammo, ehtimollar nazariyasi matematik ilm sifatida murakkab tasodifiy jarayonlarning o‘rganishdan emas, balki eng sodda qimor o‘yinlarini tahlil qilish natijasida yuzaga kela boshlagan. Shu boisdan ehtimollar nazariyasining paydo bo‘lishi XVII asr ikkinchi yarmiga mos keladi va u Paskal (1623-1662), Ferma (1601-1665) va Gyuygens (1629-1695) kabi olimlarning qimor o‘yinlarini nazariyasidagi tadqiqotlari bilan bog‘liqdir. Ehtimollar nazariyasi rivojidagi katta qadam Yakov Bernulli (1654-1705) ilmiy izlanishlari bilan bog‘liqdir. Unga, ehtimollar nazariyasining eng muhim qonuniyati, deb hisoblanuvchi “katta sonlar qonuni” tegishlidir. Ehtimollar nazariyasi rivojidagi yana bir muhim qadam de Muavr (1667-1754) nomi bilan bog‘liqdir. Bu olim tomonidan normal qonun (yoki normal taqsimot) deb ataluvchi muhim qonuniyat mavjudligi sodda holda asoslanib berildi. Keyinchalik, ma’lum bo‘ldiki, bu qonuniyat ham, ehtimollar nazariyasida muhim rol’ o‘ynar ekan. Bu qonuniyat mavjudligini asoslovchi teoremalar “markaziy limit teoremalar” deb ataladi. Ehtimollar nazariyasi rivojlanishida katta hissa mashhur matematik Laplasga (1749-1827) ham tegishlidir. U birinchi bo‘lib ehtimollar nazariyasi asoslarini qat’iy va sistematik ravishda ta’rifladi, markaziy limit teoremasining bir formasini isbotladi (Muavr-Laplas teoremasi) va ehtimollar nazariyasining bir necha tadbiqlarini keltirdi. Ehtimollar nazariyasi rivojidagi etarlicha darajada oldinga siljish Gauss (1777-1855) nomi bilan bog‘liqdir. U normal qonuniyatga yanada umumiy asos berdi va tajribadan olingan sonli ma’lumotlarni qayta ishlashning muhim usuli – “kichik kvadratlar usuli”ni yaratdi. Puasson (1781-1840) katta sonlar qonunini umumlashtirdi va ehtimollar nazariyasini o‘q uzish masalalariga qo‘lladi. Uning nomi bilan ehtimollar nazariyasida katta rol’ o‘ynovchi taqsimot qonuni nomlangandir. XVII va XIX asrlar uchun ehtimollar nazariyasining keskin rivojlanishi va u bilan har tomonlama qiziqish xarakterlidir. Keyinchalik ehtimollar nazariyasi rivojiga Rossiya olimlari V.Ya. Bunyakovskiy (1804-1889), P.L. Chebishev (1821-1894), A.A. Markov (1856-1922), A.M. Lyapunov (1857-1918), A.Ya. Xinchin (1894-1959), V.I. Romanovskiy (1879-1954), A.N. Kolmogorov (1903-1987) va ularning shogirdlari bebaho hissa qo‘shdilar. O‘zbekistonda butun dunyoga taniqli Sarimsokov (1915-1995) va S.X. Sirojiddinov (1920-1988) larning muhim rollarini alohida ta’kidlab o‘tish joizdir.

1.2 Tasodifiy hodisalar, ularning klassifikatsiyasi

Dastlab ehtimollar nazariyasining asosiy tushunchalaridan biri "tasodifiy hodisa" tushunchasini keltiramiz. Natijasini oldindan aytib bo'lmaydigan tajriba o'tkazilayotgan bo'lsin. Bunday tajribalar ehtimollar nazariyasida tasodifiy deb ataladi.

✓ *Tasodifiy hodisa*(yoki hodisa) deb, tasodifiy tajriba natijasida ro'y berishi oldindan aniq bo'lmagan hodisaga aytiladi.

Hodisalar, odatda, lotin alifbosining bosh harflari $A, B, C, \ldots$ lar bilan belgilanadi.

✓ Tajribaning har qanday natijasi *elementar hodisa* deyiladi va ω orqali belgilanadi.

✓ Tajribaning natijasida ro'y berishi mumkin bo'lgan barcha elementar hodisalar to'plami *elementar hodisalar fazosi* deyiladi va Ω orqali belgilanadi.

1.1-misol. Tajriba nomerlangan kub(o'yin soqqasi)ni tashlashdan iborat bo'lsin. U holda tajriba 6 elementar hodisadan hodisalar $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6$ lardan iborat bo'ladi. ω_i hodisa tajriba natijasida $i\,(i = 1,2,3,4,5,6)$ ochko tushishini bildiradi. Bunda elementar hodisalar fazosi: $\Omega = \{1,2,3,4,5,6\}$.

✓ Tajriba natijasida albatta ro'y beradigan hodisaga *muqarrar hodisa* deyiladi.

Elementar hodisalar fazosi muqarrar hodisaga misol bo'la oladi.

Aksincha, umuman ro'y bermaydigan hodisaga mumkin bo'lmagan hodisa deyiladi va u $\varnothing$ orqali belgilanadi.

1.1-misolda keltirilgan tajriba uchun quyidagi hodisalarni kiritamiz:

A={5 raqam tushishi};

B={juft raqam tushishi};

C={7 raqam tushishi};

D={butun raqam tushishi};

Bu yerda A va B hodisalar tasodifiy, C hodisa mumkin bo'lmagan va *D* hodisa muqarrar hodisalar bo'ladi.

1.3 Hodisalar ustida amallar

Tasodifiy hodisalar orasidagi munosabatlarni keltiramiz:

✓ A va B *hodisalar yig'indisi* deb, A va B hodisalarning kamida bittasi(ya'ni yoki A, yoki B, yoki A va B birgalikda) ro'y berishidan iborat $C = A \cup B$ ($C = A + B$) hodisaga aytiladi.

A va B *hodisalar ko'paytmasi* deb, A va B hodisalar ikkilasi ham(ya'ni A va B birgalikda)ro'y berishidan iborat $C = A \cap B$ ($C = A \cdot B$)hodisaga aytiladi.

A hodisadan B *hodisaning ayirmasi* deb, A hodisa ro'y berib, B hodisa ro'y bermasligidan iborat $C = A \setminus B$($C = A - B$) hodisaga aytiladi.

✓ A hodisaga *qarama-qarshi* $\overline{A}$ hodisa faqat va faqat A hodisa ro'y bermaganda ro'y beradi(ya'ni $\overline{A}$ hodisa A hodisa ro'y bermaganda ro'y beradi). $\overline{A}$ ni A uchun teskari hodisa deb ham ataladi.

✓ Agar A hodisa ro'y berishidan B hodisaning ham ro'y berishi kelib chiqsa A hodisa B hodisani *ergashtiradi* deyiladi va $A \subseteq B$ ko'rinishida yoziladi.

✓ Agar $A \subseteq B$ va $B \subseteq A$ bo'lsa, u holda A va B hodisalar *teng*(*teng kuchli*) hodisalar deyiladi va $A = B$ ko'rinishida yoziladi.

1.2-misol. A,B va C-ixtiyoriy hodisalar bo'lsin. Bu hodisalar orqali quyidagi hodisalarni ifodalang: *D*={uchchala hodisa ro'y berdi}; *E*={bu hodisalarning kamida bittasi ro'y berdi}; *F*={bu hodisalarning birortasi ham ro'y bermadi}; *G*={bu hodisalarning faqat bittasi ro'y berdi}.

Hodisalar ustidagi amallardan foydalanamiz: $D = A \cap B \cap C (D = A \cdot B \cdot C)$; $E = A + B + C$; $F = \overline{A} \cdot \overline{B} \cdot \overline{C}$; $G = A \cdot \overline{B} \cdot \overline{C} + \overline{A} \cdot \overline{B} \cdot C + \overline{A} \cdot B \cdot \overline{C}$.

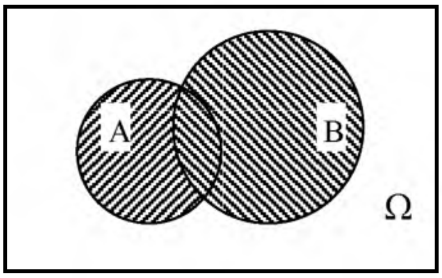
Demak hodisalarni to'plamlar kabi ham talqin etish mumkin ekan.

Belgilash	To'plamlar nazariyasidagi talqini	Ehtimollar nazariyasidagi talqini
Ω	Fazo (asosiy to'plam)	Elementar hodisalar fazosi, muqarrar hodisa
$\omega, \quad \omega \in \Omega$	ω fazo elementlari	ω elementar hodisa
$A, \quad A \subseteq \Omega$	A to'plam	A hodisa
$A \cup B, A + B$	A va B to'plamlarning yig'indisi, birlashmasi	A va B hodisalar yig'indisi (A va B ning kamida biri ro'y berishidan iborat hodisa)
$A \cap B, A \cdot B$	A va B to'plamlarning kesishmasi	A va B hodisalar ko'paytmasi (A va B ning birgalikda ro'y berishidan iborat hodisa)
$A \setminus B, A - B$	A to'plamdan B to'plamning ayirmasi	A hodisadan B hodisaning ayirmasi(A ning ro'y berishi, B ning ro'y bermasligidan iborat hodisa)
$\varnothing$	Bo'sh to'plam	Mumkin bo'lmagan hodisa
$\overline{A}$	A to'plamga to'ldiruvchi	A hodisaga teskari hodisa(A

		ning ri'y bermasligidan iborat)
$A \cap B = \varnothing$, $A \cdot B = \varnothing$	A va B to'plamlar kesishmaydi	A va B hodisalar birgalikda emas
$A \subseteq B$	A to'plam B ning qismi	A hodisa B ni ergashtiradi
$A = B$	A va B to'plamlar ustma-ust tushadi	A va B hodisalar teng kuchli

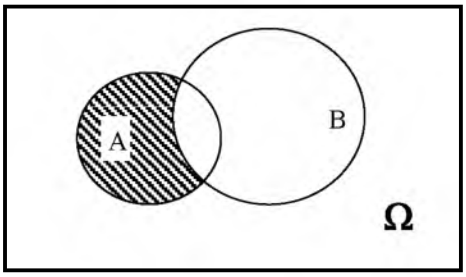
Hodisalar va ular ustidagi amallarni Eyler-Venn diarammalari yordamida tushuntirish(tasavvur qilish) qulay. Hodisalar ustidagi amallarni 1-5 rasmlardagi shakllar kabi tasvirlash mumkin.

$A + B$



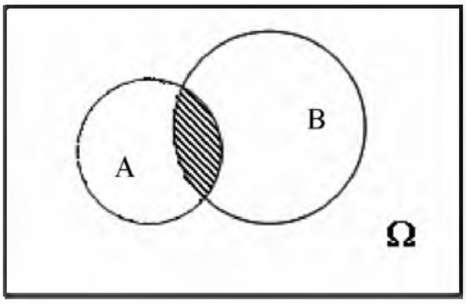
1-rasm.

A-B



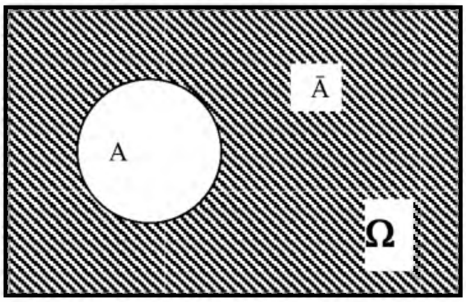
2-rasm.

$A \cdot B$



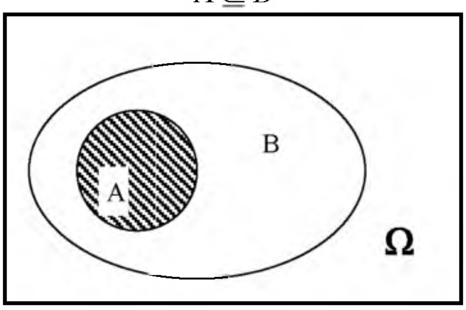
3-rasm.

$\overline{A}$



4-rasm.

$A \subseteq B$



5-rasm.

Hodisalar ustidagi amallar quyidagi xossalarga ega:

- $A+B=B+A, \quad A\cdot B=B\cdot A$;
- $(A+B)\cdot C=A\cdot C+B\cdot C,$;
- $(A+B)+C=A+(B+C), \quad (A\cdot B)\cdot C=A\cdot(B\cdot C)$;
- $A+A=A, \quad A\cdot A=A$;
- $\mathrm{A}+\Omega=\Omega, \quad A\cdot\Omega=A \quad A+\varnothing=A, \quad A\cdot\varnothing=\varnothing$;
- $A+\overline{A}=\Omega, \quad A\cdot\overline{A}=\varnothing$;
- $\overline{\varnothing}=\Omega, \quad \overline{\Omega}=\varnothing, \quad \overline{\overline{A}}=A$;
- $A-B=A\cdot\overline{B}$;
- $\overline{A+B}=\overline{A}\cdot\overline{B}$ va $\overline{A\cdot B}=\overline{A}+\overline{B}$ - de Morgan ikkilamchilik prinsipi.

1.3-misol.

a) $(A+B)\cdot(A+\overline{B})$ ifodani soddalashtiring.

Yuqoridagi xossalardan foydalanamiz:

$$(A+B)\cdot(A+\overline{B})=A\cdot A+A\cdot\overline{B}+B\cdot A+B\cdot\overline{B}=A+A\cdot(\overline{B}+B)+\varnothing=A+A\cdot\Omega=A+A=A$$

Demak, $(A+B)\cdot(A+\overline{B})=A$ ekan.

b) $A+B=A+\overline{A}\cdot B$ formulani isbotlang.

$$A+B=(A+B)\cdot\Omega=A\cdot\Omega+B\cdot\Omega=A\cdot\Omega+B\cdot(A+\overline{A})=A\cdot\Omega+(A+\overline{A})\cdot B=$$

$$=A\cdot\Omega+A\cdot B+A\cdot\overline{B}=(\Omega+B)\cdot A+A\cdot\overline{B}=\Omega\cdot A+A\cdot\overline{B}=A+A\cdot\overline{B}.$$

1.4 Tasodifiy hodisalar. Hodisalar algebrasi

Ehtimollar nazariyasining asosiy tushunchalarini keltiramiz.

Natijasi tasodifiy bo`lgan biror tajriba o`tkazilayotgan bo`lsin. $\Omega$-tajriba natijasida ro`y berishi mumkin bo`lgan barcha elementar hodisalar to`plami elementar hodisalar fazosi deyiladi; tajribaning natijasi ω esa elementar hodisa deyiladi.

✓ Agar Ω chekli yoki sanoqli to`plam bo`lsa (ya`ni elementlarini natural sonlar yordamida nomerlash mumkin bo`lsa), u holda uning ixtiyoriy qism to`plami A tasodifiy hodisa (yoki hodisa) deyiladi: $\mathrm{A}\subseteq\Omega$.

Ω to`plamdagi $\mathrm{A}$ qism to`plamga tegishli elementar hodisalar A hodisaga qulaylik yaratuvchi hodisalar deyiladi.

✓ Ω to`plam muqarrar hodisa deyiladi. $\varnothing$-bo`sh to`plam mumkin bo`lmagan hodisa deyiladi.

S-Ω ning qism to`plamlaridan tashkil topgan sistema bo`lsin.

✓ Agar

1. $\varnothing \in S$, $\Omega \in S$;
2. $A \in S$ munosabatdan $\overline{A} \in S$ kelib chiqsa;
3. $A \in S$ va $B \in S$ munosabatdan $A + B \in S, A \cdot B \in S$ kelib chiqsa S sistema algebra tashkil etadi deyiladi.
Ta'kidlash joizki, $A + B = \overline{\overline{A} \cdot \overline{B}}$, $A \cdot B = \overline{\overline{A} + \overline{B}}$ ekanligidan 3 shartdagi $A + B \in S$ va $A \cdot B \in S$ munosabatlardan ixtiyoriy bittasini talab qilish yetarlidir.

1.4-misol. $S = \{\varnothing, \Omega\}$ sistema algebra tashkil etadi: $\varnothing + \Omega = \Omega$, $\varnothing \cdot \Omega = \varnothing$, $\overline{\varnothing} = \Omega$, $\overline{\Omega} = \varnothing$.

Agar 3 shart o`rniga quyidagilarni talab qilsak $A_n \in S$, $n = 1,2,\ldots$, munosabatdan $\bigcup_{n=1}^{\infty} A_n \in S$, $\bigcap_{n=1}^{\infty} A_n \in S$ kelib chiqsa S sistema σ-algebra deyiladi.
Agar Ω chekli yoki sanoqli bo'lsa, Ω-to`plamning barcha qism to`plamlaridan tashkil topgan hodisalar sistemasi algebra tashkil etadi.

1.5 Ehtimollikning statistik ta'rifi

A hodisa n ta bog'liqsiz tajribalarda n_A marta ro'y bersin. n_A son A hodisaning chastotasi, $\frac{n_A}{n}$ munosabat esa A hodisaning nisbiy chastotasi deyiladi.

Nisbiy chastotaning statistik turg'unlik xossasi deb ataluvchi xossasi mavjud, ya'ni tajribalar soni oshishi bilan nisbiy chastotasi ma'lum qonuniyatga ega bo'ladi va biror son atrofida tebranib turadi.

Misol sifatida tanga tashlash tajribasini olaylik. Tanga A={Gerb} tomoni bilan tushishi hodisasini qaraylik. Byuffon va K.Pirsonlar tomonidan o'tkazilgan tajribalar natijasi quyidagi jadvalda keltirilgan:

Tajriba o'tkazuvchi	Tajribalar soni, n	Tushgan gerblar soni, n_A	Nisbiy chastota, n_A/n
Byuffon	4040	2048	0.5080
K.Pirson	12000	6019	0.5016
K.Pirson	24000	12012	0.5005

Jadvaldan ko'rinadiki, n ortgani sari n_A/n nisbiy chastota $\frac{1}{2} = 0.5$ ga yaqinlashar ekan.

✓ Agar tajribalar soni etarlicha ko‘p bo‘lsa va shu tajribalarda biror A hodisaning nisbiy chastotasi biror o‘zgarmas son atrofida tebransa, bu songa A hodisaning *statistik ehtimolligi* deyiladi.

A hodisaning ehtimolligi $P(A)$ simvol bilan belgilanadi. Demak,

$\lim_{n\to\infty}\frac{n_A}{n}=P(A)$ yoki yetarlicha katta n lar uchun $\frac{n_A}{n}\approx P(A)$.

Statistik ehtimollikning kamchiligi shundan iboratki, bu yerda statistik ehtimollik yagona emas. Masalan, tanga tashlash tajribasida ehtimollik sifatida nafaqat 0.5, balki 0.49 yoki 0.51 ni ham olishimiz mumkin. Ehtimollikni aniq hisoblash uchun katta sondagi tajribalar o‘tkazishni talab qiladi, bu esa amaliyotda ko‘p vaqt va xarajatlarni talab qiladi.

Statistik ehtimollik quyidagi xossalarga ega:

1. $0\le P(A)\le 1$;
2. $P(\varnothing)=0$;
3. $P(\Omega)=1$;
4. $A\cdot B=\varnothing$ bo‘lsa, u holda $P(A+B)=P(A)+P(B)$;

Isboti. 1) Ihtiyoriy A hodisaning chastotasi uchun $0\le n_A\le n\Rightarrow 0\le\frac{n_A}{n}\le 1$.

Etarlicha katta n lar uchun $\frac{n}{n_A}\approx P(A)$ bo‘lgani uchun $0\le P(A)\le 1$ bo‘ladi.

2) Mumkin bo‘lmagan hodisa uchun n_A=0.

3) Muqarrar hodisaning chastotasi n_A=n.

4) Agar $A\cdot B=\varnothing$ bo‘lsa, u holda $n_{A+B}=n_A+n_B$ va

$$P(A+B)\approx\frac{n_{A+B}}{n}=\frac{n_A+n_B}{n}=\frac{n_A}{n}+\frac{n_B}{n}\approx P(A)+P(B).$$ ■

1.6 Ehtimollikning klassik ta’rifi

Ω chekli n ta teng imkoniyatli elementar hodisalardan tashkil topgan bo‘lsin.

✓ A hodisaning ehtimolligi deb, A hodisaga qulaylik yaratuvchi elementar hodisalar soni k ning tajribadagi barcha elementar hodisalar soni n ga nisbatiga aytiladi.

$$P(A)=\frac{N(A)}{N(\Omega)}=\frac{k}{n} \qquad (1.6.1)$$

Klassik ta'rifdan foydalanib, ehtimollik hisoblashda kombinatorika elementlaridan foydalaniladi. Shuning uchun kombinatorikaning ba'zi elementlari keltiramiz. Kombinatirikada qo'shish va ko'paytirish qoidasi deb ataluvchi ikki muhim qoida mavjud.

$A=\{a_1,a_2,...,a_n\}$ va $B=\{b_1,b_2,...,b_m\}$ chekli to'plamlar berilgan bo'lsin.

✓ *Qo'shish qoidasi:* agar A to'plam elementlari soni n va B to'plam elementlari soni m bo'lib, $A\cdot B=\varnothing$(A va B to'plamlar kesishmaydigan) bo'lsa, u holda $A+B$ to'plam elementlari soni $n+m$ bo'ladi.

✓ *Ko'paytirish qoidasi:* A va B to'plamlardan tuzilgan barcha (a_i,b_j) juftliklar to'plami $C=\{(a_i,b_j): i=\overline{1,n},\ j=\overline{1,m}\}$ ning elementlari soni $n\cdot m$ bo'ladi.

n ta elementdan m ($0<m\le n$)tadan tanlashda ikkita sxema mavjud: qaytarilmaydigan va qaytariladigan tanlashlar. Birinchi sxemada olingan elementlar qayta olinmaydi(orqaga qaytarilmaydi), ikkinchi sxemada esa har bir olingan element har qadamda o'rniga qaytariladi.

I. Qaytarilmaydigan tanlashlar sxemasi

✓ *Guruhlashlar soni:* n ta elementdan m ($0<m\le n$)tadan guruhlashlar soni quyidagi formula orqali hisoblanadi:

$$C_n^m=\frac{n!}{m!(n-m)!} \qquad (1.6.2)$$

C_n^m sonlar Nyuton binomi formulasining koeffisientlaridir:

$$(p+q)^n=p^n+C_n^1p^{n-1}q+C_n^2p^{n-2}q^2+...+q^n.$$

✓ *O'rinlashtirishlar soni:* n ta elementdan m ($0<m\le n$) tadan o'rinlashtirishlar soni quyidagi formula orqali hisoblanadi:

$$A_n^m=\frac{n!}{(n-m)!}. \qquad (1.6.3)$$

✓ *O'rin almashtirishlar soni:* n ta elementdan n tadan o'rinlashtirish o'rin almashtirish deyiladi va u quyidagicha hisoblanadi:

$$P_n=n!. \qquad (1.6.4)$$

O‘rin almashtirish o‘rinlashtirishning xususiy holidir, chunki agar (1.6.3.)da $n=m$ bo‘lsa $A_n^m=\frac{n!}{(n-m)!}=\frac{n!}{0!}=n!$ bo‘ladi.

II. Qaytariladigan tanlashlar sxemasi

✓ *Qaytariladigan guruhlashlar soni:* n ta elementdan m ($0<m\le n$) tadan qaytariladigan guruhlashlar soni quyidagi formula orqali hisoblanadi:

$$\overline{C}_n^m=C_{n+m-1}^m \qquad (1.6.5)$$

✓ *Qaytariladigan o‘rinlashtirishlar soni:* n ta elementdan m ($0<m\le n$) tadan qaytariladigan o‘rinlashtirishlari soni quyidagi formula orqali hisoblanadi:

$$\overline{A}_n^m=n^m. \qquad (1.6.6)$$

✓ *Qaytariladigan o‘rin almashtirishlar soni:* k hil n ta elementdan iborat to‘plamda 1-element n_1 marta, 2-element n_2 marta,…, k- element n_k marta qaytarilsin va $n_1+n_2+...+n_k=n$ bo‘lsin, u holda n ta elementdan iborat o‘rin almashtirish $P_n(n_1,n_2,...,n_k)$ orqali belgilanadi va u quyidagicha hisoblanadi:

$$P_n(n_1,n_2,...,n_k)=\frac{n!}{n_1!n_2!...n_k!}. \qquad (1.6.4)$$

Endi ehtimollik hisoblashga doir misollar keltiramiz.

1.5-misol. Telefon nomerini terayotganda abonent oxirgi ikki raqamni eslay olmadi. U bu raqamlar har xil ekanligini eslab, ularni tavakkaliga terdi. Telefon nomeri to‘g‘ri terilganligi ehtimolligini toping.

Oxirgi ikki raqamni A_{10}^2 usul bilan terish mumkin. A={telefon nomeri to‘g‘ri terilgan} hodisasini kiritamiz. A hodisa faqat bitta elementdan iborat bo‘ladi(chunki kerakli telefon nomeri bitta bo‘ladi). Shuning uchun klassik ta’rifga ko‘ra $P(A)=\frac{N(A)}{N(\Omega)}=\frac{1}{A_{10}^2}=\frac{1}{10\cdot 9}=\frac{1}{90}\approx 0.011$.

1.6-misol. 100 ta lotoreya biletlarlaridan bittasi yutuqli bo‘lsin. Tavakkaliga olingan 10 lotoreya biletlari ichida yutuqlisi bo‘lishi ehtimolligini toping.

100 ta lotoreya biletlaridan 10 tasini C_{100}^{10} usul bilan tanlash mumkin. B={10 lotoreya biletlari ichida yutuqlisi bo'lishi } hodisasi bo'lsa, $N(B)=C_1^1\cdot C_{99}^9$ va $P(B)=\frac{N(B)}{N(\Omega)}=\frac{C_1^1\cdot C_{99}^9}{C_{100}^{10}}=\frac{1}{10}=0.1$.

1.7-misol. Pochta bo'limida 6 xildagi otkritka bor. Sotilgan 4 ta otkritkadan: a) 4 tasi bir xilda; b) 4 tasi turli xilda bo'lishi ehtimolliklarini toping.

6 xil otkritkadan 4 tasini $\overline{C_6^4}$ usul bilan tanlash mumkin. a) A={4 ta bir xildagi otkritka sotilgan} hodisasi bo'lsin. A hodisaning elementar hodisalari soni otkritkalar xillari soniga teng, ya'ni $N(A)$=6. Klassik ta'rifga ko'ra $P(A)=\frac{N(A)}{N(\Omega)}=\frac{6}{\overline{C_6^4}}=\frac{6}{126}=\frac{1}{21}$ bo'ladi. b) B={4 ta har xil otkritka sotilgan} hodisasi bo'lsin, u holda $N(B)=C_6^4$ ga teng va $P(B)=\frac{N(B)}{N(\Omega)}=\frac{C_6^4}{\overline{C_6^4}}=\frac{15}{126}=\frac{5}{42}$.

Klassik ehtimollik quyidagi xossalarga ega:

1. $P(\varnothing)=0$;
2. $P(\Omega)=1$;
3. $0\le P(A)\le 1$;
4. Agar $A\cdot B=\varnothing$ bo'lsa, u holda $P(A+B)=P(A)+P(B)$;
5. $\forall A,B\in\Omega$ uchun $P(A+B)=P(A)+P(B)-P(A\cdot B)$

Isboti. 1) $N(\varnothing)=0$ bo'lgani uchun klassik ta'rifga ko'ra $P(\varnothing)=\frac{N(\varnothing)}{N(\Omega)}=0$.

2) Klassik ta'rifga ko'ra $P(\Omega)=\frac{N(\Omega)}{N(\Omega)}=1$.

3) Ihtiyoriy A hodisa uchun $\varnothing\subseteq A\subseteq\Omega$ ekanligidan $0\le P(A)\le 1$ bo'ladi.

4) Agar $A\cdot B=\varnothing$ bo'lsa, u holda $N(A+B)=N(A)+N(B)$ va $P(A+B)=\frac{N(A+B)}{N(\Omega)}=\frac{N(A)+N(B)}{N(\Omega)}=\frac{N(A)}{N(\Omega)}+\frac{N(B)}{N(\Omega)}=P(A)+P(B)$.

5) $A+B$ va B hodisalarni birgalikda bo'lmagan ikki hodisalar yig'ndisi shaklida yozib olamiz: $A+B=A+B\cdot\overline{A}\ (1.3-misol)$, $B=B\cdot\Omega=B\cdot(A+\overline{A})=A\cdot B+B\cdot\overline{A}$, u holda 4-xossaga ko'ra $P(A+B)=P(A)+P(B\cdot\overline{A})$ va $P(B)=P(A\cdot B)+P(B\cdot\overline{A})$. Bu ikki tenglikdan $P(A+B)=P(A)+P(B)-P(A\cdot B)$ kelib chiqadi. ■

1.7 Ehtimollikning geometrik ta'rifi

Ehtimolning klassik ta'rifiga ko'ra Ω - elementar hodisalar fazosi chekli bo'lgandagina hisoblashimiz mumkin. Agar Ω cheksiz teng imkoniyatli elementar hodisalardan tashkil topgan bo'lsa, geometrik ehtimollikdan foydalanamiz.

6-rasm.

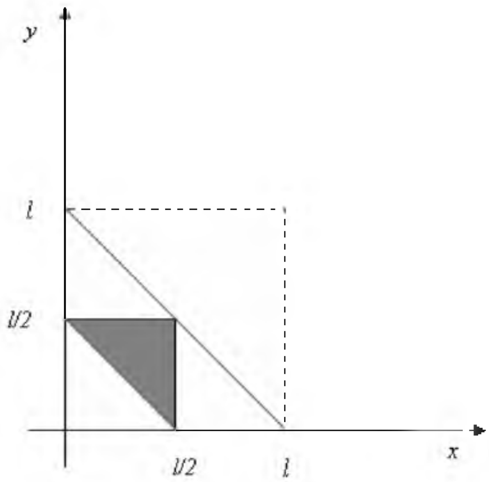
O'lchovli biror G soha berilgan bo'lib, u D sohani o'z ichiga olsin. G sohaga tavakkaliga tashlangan X nuqtani D sohaga tushishi ehtimolligini hisoblash masalasini ko'ramiz. Bu yerda X nuqtaning G sohaga tushishi muqarrar va D sohaga tushishi tasodifiy hodisa bo'ladi. $A=\{X\in D\}$ -X nuqtaning D sohaga tushishi hodisasi bo'lsin.

✓ A hodisaning geometrik ehtimolligi deb, D soha o'lchovini G soha o'lchoviga nisbatiga aytiladi, ya'ni

$$P(A)=\frac{mes\{D\}}{mes\{G\}},$$

bu yerda *mes* orqali uzunlik, yuza, hajm belgilangan.

1.8-misol. l uzunlikdagi sterjen tavakkaliga tanlangan ikki nuqtada bo'laklarga bo'lindi. Hosil bo'lgan bo'laklardan uchburchak yasash mumkin bo'lishi ehtimolligini toping.



7-rasm.

Birinchi bo'lak uzunligini *x*, ikkinchi bo'lak uzunligini *y* bilan belgilasak, uchinchi bo'lak uzunligi *l-x-y* bo'ladi. Bu yerda $\Omega=\{(x,y):0<x+y<l\}$, ya'ni $0<x+y<l$ sterjenning bo'laklari uzunliklarining barcha bo'lishi mumkin bo'lgan kombinatsiyasidir. Bu bo'laklardan uchburchak yasash mumkin bo'lishi uchun quyidagi shartlar bajarilishi kerak: $x+y>l-x-y$, $x+l-x-y>y$, $y+l-x-y>x$.

Bulardan $x < \frac{l}{2}, \quad y < \frac{l}{2}, \quad x + y > \frac{l}{2}$ ekanligi kelib chiqadi.

Bu tengsizliklar 7-rasmdagi bo‘yalgan sohani bildiradi. Ehtimollikning geometrik ta’rifiga ko‘ra:

$$P(A) = \frac{mes\{A\}}{mes\{G\}} = \frac{\frac{1}{2} \cdot \frac{l}{2} \cdot \frac{l}{2}}{\frac{1}{2} \cdot l \cdot l} = \frac{1}{4}.$$

1.9-misol. (Uchrashuv haqida)

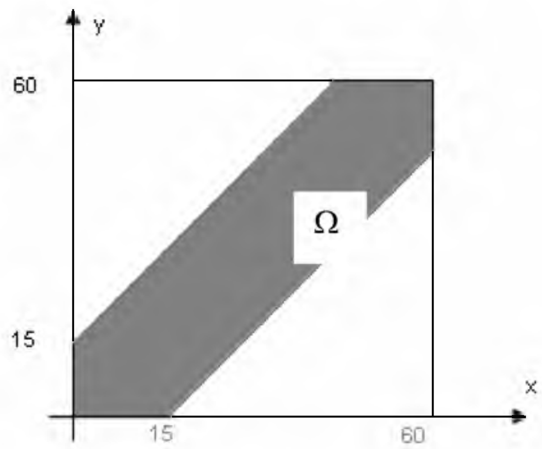
Ikki do‘st soat 9 bilan 10 orasida uchrashishga kelishishdi. Birinchi kelgan kishi do‘stini 15 daqiqa davomida kutishini, agar shu vaqt mobaynida do‘sti kelmasa u ketishi mumkinligini shartlashib olishdi. Agar ular soat 9 bilan 10 orasida ixtiyoriy momentda kelishlari mumkin bo‘lsa, bu ikki do‘stning uchrashishi ehtimolini toping.

Birinchi kishi kelgan momentni x, ikkinchisinikini y bo‘lsin: $0 \le x \le 60$, $0 \le y \le 60$ U holda ularning uchrashishlari uchun $|x - y| \le 15$ tengsizlik bajarilishi kerak.

Demak, $\Omega = \{(x, y) : 0 \le x \le 60, 0 \le y \le 60\}$, $A = \{(x, y) : |x - y| \le 15\}$. x va y larni Dekart koordinatalar tekisligida tasvirlaymiz(8-rasm).

U holda

$$P(A) = \frac{mes\{A\}}{mes\{G\}} = \frac{60^2 - 2 \cdot \frac{1}{2} \cdot 45 \cdot 45}{60^2} = \frac{7}{16}.$$



8-rasm.

1.8 Ehtimollikning aksiomatik ta’rifi

Ehtimollar nazariyasini aksiomatik qurishda A.N. Kolmogorov tomonidan 30-yillarning boshlarida asos solingan.

Ω- biror tajribaning barcha elementar hodisalar to‘plami, S-hodisalar algebrasi bo‘lsin.

✓ S hodisalar algebrasida aniqlangan, haqiqiy qiymatlar qabul qiluvchi $P(A)$ fuksiya ehtimollik deyiladi, agar u uchun quyidagi aksiomalar o‘rinli bo‘lsa:

A1: ihtiyoriy $A \in S$ hodisaning ehtimolligi manfiy emas $P(A) \geq 0$ (nomanfiylik aksiomasi);

A2: muqarrar hodisaning ehtimolligi birga teng $P(\Omega) = 1$ (normallashtirish aksiomasi);

A3: juft-jufti bilan birgalikda bo‘lmagan hodisalar yig‘indisining ehtimolligi shu hodisalar ehtimollari yig‘indisiga teng, ya’ni agar $A_i \cdot A_j = \varnothing, \quad i \neq j$ bo‘lsa, u holda

$$P(\bigcup_k A_k) = \sum_k P(A_k)$$

(additivlik aksiomasi);

(Ω, S, P) uchlik ehtimollik fazosi deyiladi, bu yerda Ω-elementar hodisalar fazosi, S-hodisalar algebrasi, P- A1-A3 aksiomalarni qanoatlantiruvchi sanoqli funksiya.

1.9 Ehtimollikning xossalari

Kolmogorov aksiomalarining tatbiqi sifatida quyidagi xossalarni keltiramiz:

1. Mumkin bo‘lmagan hodisaning ehtimoli nolga teng

$$P(\varnothing) = 0 .$$

2. Qarama-qarshi hodisalarning ehtimolliklari yig‘indisi birga teng

$$P(A) + P(\overline{A}) = 1 .$$

3. Ixtiyoriy hodisaning ehtimolligi uchun quyidagi munosabat o‘rinli:

$$0 \leq P(A) \leq 1$$

4. Agar $A \subseteq B$ bo‘lsa, u holda $P(A) \leq P(B)$.

5. Agar birgalikda bo‘lmagan $A_1, A_2, ..., A_n$ hodisalar to‘la gruppani tashkil etsa, ya’ni $\bigcup_{i=1}^{n} A_i = \Omega$ va $A_i \cdot A_j = \varnothing, \quad i \neq j$ bo‘lsa u holda

$$\sum_{i=1}^{n} P(A_i) = 1 .$$

Isboti:

1. $A+\varnothing=A,\quad A\cdot\varnothing=\varnothing$ tengliklardan A3 aksiomaga ko‘ra $P(A)+P(\varnothing)=P(A)\Rightarrow P(\varnothing)=0$

2. $A+\overline{A}=\Omega\quad A\cdot\overline{A}=\Omega$ tengliklardan $P(A)+P(\overline{A})=P(\Omega)$ hamda A2 va A3 aksiomalardan esa $P(A)+P(\overline{A})=1$ tenglik kelib chiqadi.

3. 2-xossaga ko‘ra $P(A)=1-P(\overline{A})$ va A1 aksiomaga asosan $0\le P(A)\le 1$.

4. $A\subseteq B$ ekanligidan $B=(B-A)+A$ va $(B-A)A=\varnothing$. A3 aksiomaga ko‘ra $P(B)=P(B-A)+P(A)$, ammo $P(B-A)\ge 0$ bo‘lgani uchun $P(A)\le P(B)$.

5. $A_1+A_2+...+A_n=\Omega$ tenglik, A2 va A3 aksiomalarga ko‘ra $P(A_1+A_2+...+A_n)=P(A_1)+P(A_2)+...+P(A_n)$. ■

1.10 Ehtimolliklar fazosi

Elementar hodisalar fazosi cheksiz bo‘lsin: $\Omega=\{\omega_1,\omega_2,...,\omega_n,...\}$. S esa Ω ning barcha qism to‘plamlaridan tashkil topgan hodisalar algebrasi bo‘lsin. Har bir $\omega_i\in\Omega,\ \ i=1,2,...$ elementar hodisaga $p(\omega_i)$ sonni mos qo‘yamiz. $p(\omega_i)$-elementar hodisaning ehtimoli deyiladi. Demak, Ω da quyidagi shartlarni qanoatlantiruvchi sonli $p(\omega_i)$ funksiya kiritamiz:

1. $\forall\omega_i\in\Omega,\quad P(\omega_i)\ge 0$;
2. $\sum_{i=1}^{\infty}p(\omega_i)=1$.

U holda $A\in\Omega$ hodisaning ehtimolligi yig‘indi shaklida ifodalanadi:

$$P(A)=\sum_{\omega_i\in A}P(\omega_i)\qquad (1.10.1)$$

Ehtimollikni bunday aniqlash Kolmogorov aksiomalarini qanoatlantiradi:

1. $P(A)=\sum_{\omega_i\in A}P(\omega_i)\ge 0$, chunki har bir $P(\omega_i)\ge 0$;
2. $P(\Omega)=\sum_{\omega_i\in\Omega}p(\omega_i)=\sum_{i=1}^{n}p(\omega_i)=1$;
3. Agar $A\cdot B=\varnothing$ bo‘lsa, u holda

$$P(A+B)=\sum_{\omega_i\in A+B}p(\omega_i)=\sum_{\omega_i\in A}p(\omega_i)+\sum_{\omega_i\in B}p(\omega_i)=P(A)+P(B).$$

Bunday aniqlangan $\{\Omega, S, P\}$ uchlik ehtimolliklar fazosi(yoki diskret ehtimolliklar fazosi) deyiladi.

Agar $\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$ - chekli fazo va tajribadagi barcha elementar hodisalar teng imkoniyatli bo'lsa, ya'ni

$$p(\omega_1) = p(\omega_2) = \ldots = p(\omega_n) = \frac{1}{n}, \qquad (1.10.2)$$

u holda (1.10.1) formula quyidagi ko'rinishga ega bo'ladi:

$$P(A) = \sum_{\omega_i \in A} p(\omega_i) = \underbrace{\frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n}}_{m} = \frac{m}{n}. \qquad (1.10.3)$$

Bu yerda m A hodisaga tegishli elementar hodisalar soni. Bu esa ehtimollikni klassik ta'rifga ko'ra hisoblashdir. Demak, klassik ehtimol (1.10.1) formula orqali aniqlangan ehtimollikning xususiy holi ekan.

1.11 Shartli ehtimollik

A va B hodisalar biror tajribadagi hodisalar bo'lsin.

✓ B hodisaning A hodisa ro'y bergandagi *shartli ehtimolligi* deb,

$$\frac{P(A \cdot B)}{P(A)} \quad (P(A) \neq 0) \qquad (1.11.1)$$

nisbatga aytiladi. Bu ehtimollikni $P(B/A)$ orqali belgilaymiz.

Shartli ehtimollik ham Kolmogorov aksiomalarini qanoatlantiradi:

1. $P(B/A) \geq 0$;

2. $P(\Omega/A) = \frac{P(\Omega \cdot A)}{P(A)} = \frac{P(A)}{P(A)} = 1$;

3. Agar $B \cdot C = \varnothing$ bo'lsa, u holda

$$P((B+C)/A) = \frac{P((B+C)\cdot A)}{P(A)} = \frac{P(B \cdot A + C \cdot A)}{P(A)} = \frac{P(B \cdot A) + P(C \cdot A)}{P(A)} =$$

$$= \frac{P(B \cdot A)}{P(A)} + \frac{P(C \cdot A)}{P(A)} = P(B/A) + P(C/A),$$

chunki $B\cdot C=\varnothing$ ekanligidan, $(B\cdot A)\cdot(C\cdot A)=B\cdot A\cdot A\cdot C=B\cdot C\cdot A=\varnothing\cdot A=\varnothing$

1.10-misol. Idishda 3 ta oq va 7 ta qora shar bor. Tavakkaliga ketma-ket bittadan 2 ta shar olinadi. Birinchi shar oq rangda bo‘lsa ikkinchi sharning qora rangda bo‘lishi ehtimolligini toping.

Bu misolni ikki usul bilan yechish mumkin:

1) A={birinchi shar oq rangda}, B={ikkinchi shar qora rangda}. A hodisa ro‘y berganidan so‘ng idishda 2 ta oq va 7 ta qora shar qoladi. Shuning uchun $P(B/A)=\frac{7}{9}$.

2) (1.11.1) formuladan foydalanib, hisoblaymiz: $P(A)=\frac{3}{10}$, $P(AB)=\frac{3}{10}\cdot\frac{7}{9}=\frac{7}{30}$

Shartli ehtimollik formulasiga ko‘ra: $P(B/A)=\frac{P(A\cdot B)}{P(A)}=\frac{7/30}{3/10}=\frac{7}{9}$.

Shartli ehtimollik formulasidan hodisalar ko‘paytmasi ehtimolligi uchun ushbu formula kelib chiqadi:

$$P(A\cdot B)=P(A)\cdot P(B/A)=P(B)\cdot P(A/B) \qquad (1.11.2)$$

(1.11.2) tenglik ko‘paytirish qoidasi(teoremasi) deyiladi. Bu qoidani n ta hodisa uchun umumlashtiramiz:

$$P(A_1\cdot A_2\cdot\ldots\cdot A_n)=P(A_1)\cdot P(A_2/A_1)\cdot P(A_3/A_1A_2)\ldots\cdot P(A_n/A_1A_2\ldots A_{n-1}). \qquad (1.11.3)$$

✓ Agar $P(A/B)=P(A)$ tenglik o‘rinli bo‘lsa, u holda A hodisa B hodisaga bog‘liq emas deyiladi va $A\perp B$ orqali belgilanadi.

Agar $A\perp B$ bo‘lsa, u holda (1.11.2) formulani quyidagicha yozish mumkin:

$$P(A\cdot B)=P(B)\cdot P(A/B)=P(B)\cdot P(A).$$

✓ A va B hodisalar o‘zaro bog‘liq emas deyiladi, agar

$$P(A\cdot B)=P(A)\cdot P(B)$$

munosabat o‘rinli bo‘lsa.

Lemma. Agar $A\perp B$ bo‘lsa, u holda $A\perp\overline{B}$, $\overline{A}\perp B$ va $\overline{A}\perp\overline{B}$ bo‘ladi.

Isboti: $A \perp B$ bo‘lsin. U holda $P(A \cdot B) = P(A) \cdot P(B)$ munosabat o‘rinli bo‘ladi. $P(B) + P(\overline{B}) = 1$ tenglikdan foydalanib, quyidagiga ega bo‘lamiz:

$$P(A \cdot \overline{B}) = P(A \cdot (\Omega - B)) = P(A \cdot \Omega - A \cdot B) = P(A - A \cdot B) = P(A) - P(A \cdot B) =$$
$$= P(A) - P(A) \cdot P(B) = P(A) \cdot (1 - P(B)) = P(A) \cdot P(\overline{B}).$$

Demak, $P(A \cdot \overline{B}) = P(A) \cdot P(\overline{B}) \Rightarrow A \perp \overline{B}$. Qolganlari ham xuddi shunday isbotlanadi. ■

1.12 To‘la ehtimollik va Bayes formulalari

$A_1, A_2, ..., A_n$ juft-jufti bilan birgalikda bo‘lmagan hodisalar to‘la gruppani tashkil etsin, ya’ni $\bigcup_{i=1}^{n} A_i = \Omega$ va $A_i \cdot A_j = \varnothing, \quad i \neq j$. U holda $A_1 + A_2 + ... + A_n = \Omega$ ekanligini hisobga olib, B ni $B = B \cdot \Omega = B \cdot (A_1 + A_2 + ... + A_n) = B \cdot A_1 + B \cdot A_2 + ... + B \cdot A_n$ ko‘rinishda yozamiz. $A_i \cdot A_j = \varnothing, \quad i \neq j$ ekanligidan $(B \cdot A_i) \cdot (B \cdot A_j) = \varnothing, \quad i \neq j$ ekani kelib chiqadi. B hodisaning ehtimolligini hisoblaymiz:

$$P(B) = P(B \cdot A_1 + B \cdot A_2 + ... + B \cdot A_n) =$$
$$= P(B \cdot A_1) + P(B \cdot A_2) + ... + P(B \cdot A_n). \qquad (1.12.1)$$

Ko‘paytirish qoidasiga ko‘ra $P(B \cdot A_i) = P(A_i) \cdot P(B / A_i), i = \overline{1,n}$ bo‘ladi. Bu tenglikni (1.12.1) ga qo‘llasak,

$$P(B) = P(A_1)P(B / A_1) + P(A_2)P(B / A_2) + ... + P(A_n)P(B / A_n).$$

✓ Agar $B \subset \sum_{i=1}^{n} A_i$ bo‘lsa, u holda

$$P(B) = \sum_{i=1}^{n} P(A_i)P(B / A_i) \qquad (1.12.2)$$

tenglik o‘rinli bo‘ladi. Bu tenglik *to‘la ehtimollik formulas*i deyiladi.

1.11-masala. Detallar partiyasi uch ishchi tomonidan tayyorlanadi. Birinchi ishchi barcha detallarning 25%ini, ikkinchi ishchi 35%ini, uchinchsi esa 40%ini tayyorlaydi. Bu uchchala ishchining tayyorlagan detallarining sifatsiz bo‘lish ehtimolliklari mos ravishda 0.05,0.04 va 0.02

ga teng bo‘lsa, tekshirish uchun partiyadan olingan detalning sifatsiz bo‘lish ehtimolligini toping.

A_i={detal i-ishchi tomonidan tayyorlangan} $i=\overline{1,3}$, B={tekshirish uchun olingan detal sifatsiz} hodisalarni kiritamiz va quyidagi ehtimolliklarni hisoblaymiz:

$$P(A_1)=\frac{25\%}{100\%}=0.25,\quad P(A_2)=\frac{35\%}{100\%}=0.35,\quad P(A_3)=\frac{40\%}{100\%}=0.4,$$

$P(B/A_1)=0.05,\quad P(B/A_2)=0.04,\quad P(B/A_3)=0.02$. To‘la ehtimollik formulasiga asosan $P(B)=0.25\cdot0.05+0.35\cdot0.04+0.4\cdot0.02=0.0345$.

A_i va B hodisalar ko‘paytmasi uchun

$$P(A_i\cdot B)=P(B)\cdot P(A_i/B) \qquad (1.12.3)$$

$$P(A_i\cdot B)=P(A_i)\cdot P(B/A_i) \qquad (1.12.4)$$

tengliklar o‘rinli. (1.12.3) va (1.12.4) tengliklardan quyidagilarni hosil qilamiz:

$$P(B)\cdot P(A_i/B)=P(A_i)\cdot P(B/A_i),$$

$$P(A_i/B)=\frac{P(A_i)P(B/A_i)}{P(B)}. \qquad (1.12.5)$$

Bu yerda $P(B)=\sum_{i=1}^{n}P(A_i)P(B/A_i)$. (1.12.5) tenglik *Bayes formulasi* deyiladi. Bayes formulasi yana *gipotezalar teoremasi* deb ham ataladi. Agar $A_1,A_2,...,A_n$ hodisalarni gipotezalar deb olsak, u holda $P(A_i)$ ehtimollik A_i gipotezaning aprior(“a priori” lotincha tajribagacha), $P(A_i/B)$ shartli ehtimollik esa aposterior(“a posteriori” tajribadan keyingi) ehtimolligi deyiladi.

1.12-masala. 1.11-misolda sifatsiz detal ikkinchi ishchi tomonidan tayyorlangan bo‘lishi ehtimolligini toping. Bayes formulasiga ko‘ra:

$$P(A_2/B)=\frac{0.35\cdot0.04}{0.25\cdot0.05+0.35\cdot0.04+0.4\cdot0.02}=\frac{28}{69}\approx0.4.$$

1.13 Bog‘liqsiz tajribalar ketma-ketligi. Bernulli formulasi

Agar bir necha tajribalar o‘tkazilayotganida, har bir tajribada biror A hodisaning ro‘y berish ehtimolligi boshqa tajriba natijalariga bog‘liq bo‘lmasa, bunday tajribalar bog‘liqsiz tajribalar deyiladi.

n ta bog‘liqsiz tagribalar o‘tkazilayotgan bo‘lsin. Har bir tajribada A hodisaning ro‘y berish ehtimolligi $P(A)=p$ va ro‘y bermasligi ehtimolligi $P(\overline{A})=1-p=q$ bo‘lsin.

Masalan, 1) nishonga qarata o‘q uzish tajribasini ko‘raylik. Bu yerda A={o‘q nishonga tegdi}-muvaffaqqiyat va $\overline{A}$={o‘q nishonga tegmadi}-muvaffaqqiyatsizlik; 2) *n* ta mahsulotni sifatsizlikka tekshirilayotganda A={mahsulot sifatli}-muvaffaqqiyat va $\overline{A}$={mahsulot sifatsiz}-muvaffaqqiyatsizlik bo‘ladi.

Bu kabi tajribalarda elementar hodisalar fazosi Ω faqat ikki elementdan iborat bo‘ladi: $\Omega=\{\omega_0,\omega_1\}=\{\overline{A},A\}$, bu erda ω_0-A hodisa ro‘y bermasligini, ω_1-A hodisa ro‘y berishini bildiradi. Bu hodisalarning ehtimolliklari mos ravishda *p* va *q (p+q=1)* lar orqali belgilanadi.

Agar *n* ta tajriba o‘tkazilayotgan bo‘lsa, u holda elementar hodisalar fazosining elementar hodisalari soni 2^n ga teng bo‘ladi. Masalan, $n=3$ da $\Omega=\{\omega_0,\omega_1,\ldots,\omega_7\}=\{\overline{A}\overline{A}\overline{A},\overline{A}\overline{A}A,\overline{A}A\overline{A},\overline{A}AA,A\overline{A}\overline{A},A\overline{A}A,AA\overline{A},AAA\}$, ya’ni Ω to‘plam $2^3=8$ ta elementar hodisadan iborat. Har bir hodisaning ehtimolligini ko‘paytirish teoremasiga ko‘ra hisoblash mumkin:

$$p(\omega_0)=P(\overline{A}\overline{A}\overline{A})=P(\overline{A})P(\overline{A})P(\overline{A})=q^3,$$
$$p(\omega_1)=P(\overline{A}\overline{A}A)=P(\overline{A})P(\overline{A})P(A)=pq^2,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$p(\omega_7)=P(AAA)=P(A)P(A)P(A)=p^3.$$

n ta bog‘liqsiz tajribada A hodisa *m* marta ro‘y berish ehtimolligini hisoblaylik:

$$P_n(m)=P(\underbrace{A\cdot A\cdot\ldots\cdot A}_{m ta}\cdot\underbrace{\overline{A}\cdot\overline{A}\cdot\ldots\cdot\overline{A}}_{(n-m)ta})+P(\overline{A}\underbrace{A\cdot A\cdot\ldots\cdot A}_{m ta}\cdot\underbrace{\overline{A}\cdot\overline{A}\cdot\ldots\cdot\overline{A}}_{(n-(m-1))ta})+\ldots+$$
$$P(\underbrace{\overline{A}\cdot\overline{A}\cdot\ldots\cdot\overline{A}}_{(n-(m-1))ta}\cdot\underbrace{A\cdot A\cdot\ldots\cdot A}_{m ta}\cdot\overline{A})+P(\underbrace{\overline{A}\cdot\overline{A}\cdot\ldots\cdot\overline{A}}_{(n-m)ta}\cdot\underbrace{A\cdot A\cdot\ldots\cdot A}_{m ta}).$$

Har bir qo'shiluvchi ko'paytirish teoremasiga ko'ra $p^m q^{n-m}$ ga teng. Demak,

$$P_n(m) = \underbrace{p^m q^{n-m} + p^m q^{n-m} + \ldots + p^m q^{n-m}}_{C_n^m \ ta \quad qo'shiluvchi} = C_n^m p^m q^{n-m}, \quad m = 0,1,\ldots n.$$

✓ Agar *n* ta bo'g'liqsiz tajribaning har birida *A* hodisaning ro'y berish ehtimolligi *p* ga, ro'y bermasligi *q* ga teng bo'lsa, u holda *A* hodisaning ***m*** marta ro'y berish ehtimolligi quyidagi ifodaga teng bo'ladi:

$$P_n(m) = C_n^m p^m q^{n-m}, \quad m = 0,1,\ldots n. \qquad (1.13.1)$$

(1.13.1) formula Bernulli formulasi deyiladi. $P_n(m)$ ehtimolliklar uchun $\sum_{m=0}^{n} P_n(m) = 1$ tenglik o'rinlidir. Haqiqatan ham,

$$(q+px)^n = q^n + C_n^1 q^{n-1} px + C_n^2 q^{n-2} p^2 x^2 + \ldots + p^n x^n$$

Nyuton binomi formulasida $x = 1$ deb olsak,
$(q+p)^n = q^n + C_n^1 q^{n-1} p + C_n^2 q^{n-2} p^2 + \ldots + p^n$, ya'ni
$1 = P_n(0) + P_n(1) + \ldots + P_n(n) = \sum_{m=0}^{n} P_n(m)$ bo'ladi.

(1.13.1) ehtimolliklar xossalari:

1. $\sum_{m=0}^{n} P_n(m) = 1$.

2. Agar $m_1 \le m \le m_2$ bo'lsa, $P_n(m_1 \le m \le m_2) = \sum_{m=m_1}^{m_2} P_n(m)$.

3. ***n*** ta bog'liqsiz tajribada *A* hodisaning kamida 1 marta ro'y berishi ehtimolligi $P = 1 - q^n$ bo'ladi.
Chunki, $P_n(0) + \underbrace{P_n(1) + \ldots + P_n(n)}_{P} = 1 \Rightarrow P = 1 - P_n(0) = 1 - q^n$.

4. Agar $P_n(m)$ ehtimollikning eng katta qiymati $P_n(m_0)$ bo'lsa, u holda m_0 quyidagicha aniqlanadi: $np - q \le m_0 \le (n+1)p$, m_0-eng ehtimolli son deyiladi va
a) agar *np-q* kasr son bo'lsa, u holda m_0 yagonadir;
b) agar *np-q* butun son bo'lsa, u holda m_0 ikkita bo'ladi.

1.13-misol. Ikki teng kuchli shaxmatchi shaxmat o‘ynashmoqda. Qaysi hodisaning ehtimolligi katta: 4 ta partiyadan 2 tasida yutishmi yoki 6 ta partiyadan 3 tasida yutish. Birinchi holda: n=4, m=2, $p=\frac{1}{2}$, Bernulli formulasiga ko‘ra $P_4(2)=C_4^2\left(\frac{1}{2}\right)^2\left(1-\frac{1}{2}\right)^{4-2}=6\cdot\frac{1}{2^2}\cdot\frac{1}{2^2}=\frac{6}{16}$.

Ikkinchi holda n=6, m=3, $p=\frac{1}{2}$ va Bernulli formulasiga ko‘ra $P_6(3)=C_6^3\left(\frac{1}{2}\right)^3\left(1-\frac{1}{2}\right)^{6-3}=20\cdot\frac{1}{2^3}\cdot\frac{1}{2^3}=\frac{5}{16}$. $\frac{6}{16}>\frac{5}{16}\Rightarrow P_4(2)>P_6(3)$. Demak, 4 ta partiyadan 2 tasida yutish ehtimolligi katta ekan.

1.14 Limit teoremalar

Agar n va m lar katta sonlar bo‘lsa, u holda Bernulli formulasidan foydalanib, $P_n(m)$ ehtimollikni hisoblash qiyinchilik tug‘diradi. Xuddi shunday, $p(q)$ ehtimollik juda kichik qiymatlar qabul qilsa ham qiyinchiliklarga duch kelamiz. Shu sababli, $n\to\infty$ da $P_n(m)$ uchun asimptotik(taqribiy) formulalar topish muammosini tug‘diradi.

Puasson formulasi

✓ Agar $n\to\infty$ da A hodisaning ro‘y berish ehtimolligi p har bir tajribada cheksiz kamaysa(ya’ni $np\to a>0$), u holda

$$\lim_{n\to\infty}P_n(m)=\frac{a^m\cdot e^{-a}}{m!},\ m=0,1,2,\dots\ . \qquad (1.14.1)$$

(1.14.1) formula Puassonning asimptotik formulasi deyiladi.

$p=\frac{a}{n}$ belgilash kiritib, Bernulli formulasidan

$$P_n(m)=C_n^m p^m q^{n-m}=\frac{n!}{m!(n-m)!}\cdot\left(\frac{a}{n}\right)^m\left(1-\frac{a}{n}\right)^{n-m}=$$

$$=\frac{n\cdot(n-1)\cdot\ldots\cdot(n-(m-1))}{m!}\cdot\frac{a^m}{n^m}\cdot\left(1-\frac{a}{n}\right)^n\cdot\left(1-\frac{a}{n}\right)^{-m}=$$

$$=\frac{a^m}{m!}\cdot\frac{n}{n}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n}\cdot\ldots\cdot\frac{n-(m-1)}{n}\cdot\left(1-\frac{a}{n}\right)^n\cdot\left(1-\frac{a}{n}\right)^{-m}=$$

$$=\frac{a^m}{m!}\cdot 1\cdot\left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\cdot\ldots\cdot\left(1-\frac{m-1}{n}\right)\cdot\left(1-\frac{a}{n}\right)^n\cdot\left(1-\frac{a}{n}\right)^{-m} \qquad (1.14.2)$$

$\lim\limits_{n\to\infty}\left(1-\frac{a}{n}\right)^n=e^{-a}$ ekanligini e'tiborga olib, (1.14.2) tenglikdan limitga o'tamiz:

$$\lim_{n\to\infty}P_n(m)=\frac{a^m}{m!}e^{-a}.$$

Demak, yetarlicha katta *n* larda (kichik *p* da)

$$P_n(m)\approx\frac{a^m\cdot e^{-a}}{m!},\quad a=np,\quad m=0,1,...,n \qquad (1.14.3)$$

(1.14.3) formula Puasson formulasi deyiladi. Odatda Puasson formulasidan $n\geq 50,\quad np\leq 10$ bo'lgan hollarda foydalaniladi.

1.14-misol. Telefon stansiyasi 2000 ta abonentga xizmat ko'rsatadi. Agar har bir abonent uchun unig bir soatning ichida qo'ng'iroq qilishi ehtimolligi 0.003 bo'lsa, bir soatning ichida 5 ta abonent qo'ngiroq qilishi ehtimolligini toping.

n=2000, *p*=0.003, *m*=5, *a*=*np*=2000·0.003=6<10. Demak, Puasson formulasiga ko'ra $P_{2000}(5)=\frac{6^5\cdot e^{-6}}{5!}\approx 0{,}13$.

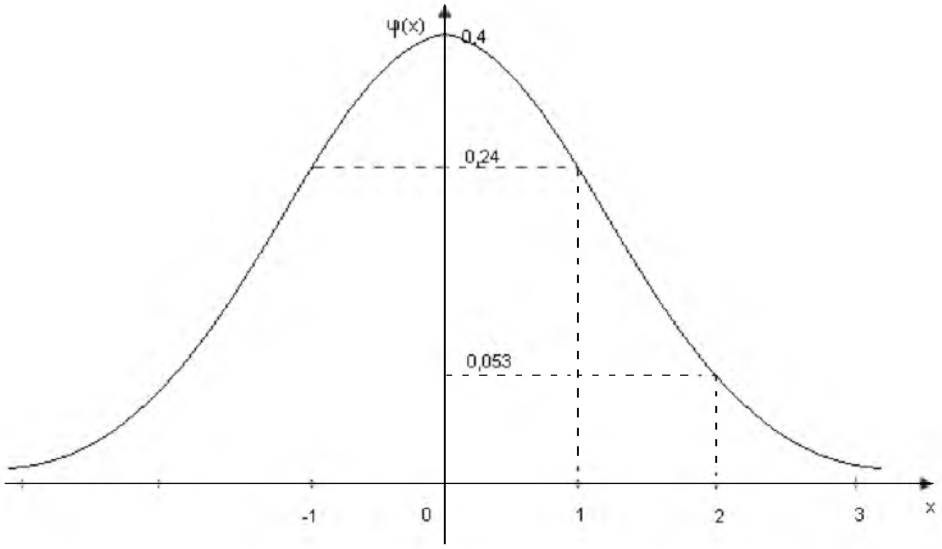
Muavr-Laplasning lokal teoremasi

Agar *p* ($p\neq 0, p\neq 1$)ehtimollik nol atrofidagi son bo'lmasa va *n* etarlicha katta bo'lsa, u holda $P_n(m)$ ehtimollikni hisoblash uchun Muavr-Laplas teoremasidan foydalanish mumkin.

Teorema(Muavr-Laplas) Agar *n* ta bog'liqsiz tajribada *A* hodisaning ro'y berish ehtimolligi $0<p<1$ bo'lsa, u holda yetarlicha katta *n* larda

$$P_n(m)\approx\frac{1}{\sqrt{npq}}\cdot\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{x^2}{2}},\ x=\frac{m-np}{\sqrt{npq}} \qquad (1.14.4)$$

-taqribiy formula o‘rinli. Bu yerda $\varphi(x)=\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{x^2}{2}}$ funksiya Gauss funksiyasi deyiladi(9-rasm).



9-rasm.

$\varphi(x)$ funksiya uchun x argument qiymatlariga mos qiymatlari jadvali tuzilgan(1-ilova). Jadvaldan foydalanayotganda quyidagilarni e’tiborga olish kerak:

1) $\varphi(x)$ funksiya juft funksiya, ya’ni $\varphi(-x)=\varphi(x)$.

2) agar $x\geq 4$ bo‘lsa, $\varphi(x)=0$ deb olish mumkin.

1.15-misol. Bitta o‘q otilganda o‘qning nishonga tegish ehtimolligi 0.7 ga teng. 200 ta o‘q otilganda nishonga 160 ta o‘q tegishi ehtimolligini toping.

Bu yerda *n*=200, *p*=0.7, *q*=1-*p*=0.3, *m*=160. (1.14.4) ga ko‘ra $\sqrt{npq}=\sqrt{200\cdot 0.7\cdot 0.3}=\sqrt{42}\approx 6.48$, $x=\frac{160-200\cdot 0.7}{\sqrt{42}}=\frac{20}{6.48}\approx 3.09$. Agar $\varphi(3.09)\approx 0.0034$ ekanligini hisobga olsak, u holda $P_{200}(160)\approx\frac{1}{6.48}\cdot 0.0034\approx 0.0005$.

Muavr-Laplasning integral teoremasi

Agar n yetarlicha katta va A hodisa n ta tajribada kamida m_1 va ko‘pi bilan m_2 marta ro‘y berish ehtimolligi $P_n(m_1\leq m\leq m_2)$ni topish talab etilsa, u holda Muavr-Laplasning integral teoremasidan foydalanish mumkin.

Teorema(Muavr-Laplas) Agar A hodisaning ro‘y berish ehtimolligi($0 < p < 1$) o‘zgarmas bo‘lsa, u holda

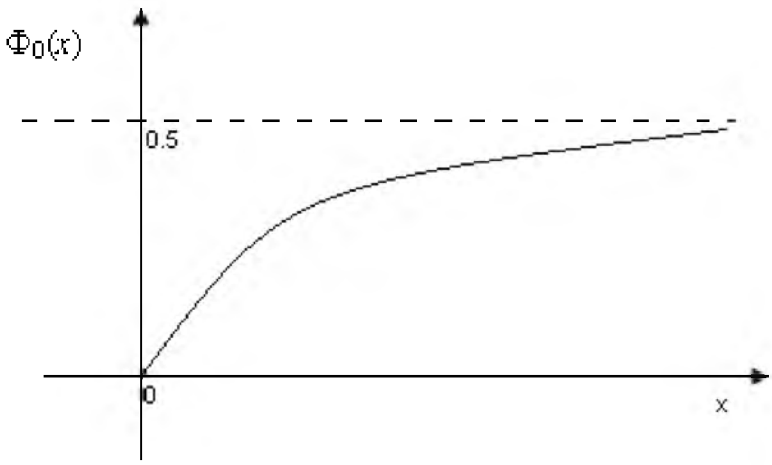
$$P_n(m_1 \le m \le m_2) \approx \frac{1}{\sqrt{2\pi}} \int_{x_1}^{x_2} e^{-x^2/2} dx, \qquad (1.14.5)$$

taqribiy formula o‘rinli, bu yerda $x_i = \frac{m_i - np}{\sqrt{npq}}, \quad i = 1,2$.

(1.14.5) formuladan foydalanilganda hisoblashlarni soddalashtirish uchun maxsus funksiya kiritiladi:

$$\Phi_0(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2} dt. \qquad (1.14.6)$$

(1.14.6)-Laplas funksiyasi deyiladi.



10-rasm.

$\Phi_0(x)$ funksiya toq funksiya:

$$\Phi_0(-x) = \frac{1}{\sqrt{2\pi}} \int_0^{-x} e^{-t^2/2} dt = [t = -z] = -\frac{1}{\sqrt{2\pi}} \int_0^x e^{-z^2/2} dz = -\Phi_0(x).$$

Agar $x \ge 5$ bo‘lsa, u holda $\Phi_0(x) = 0.5$ deb hisoblash mumkin;

$\Phi_0(x)$ funksiya grafigi 10-rasmda keltirilgan.

(1.14.5) dagi tenglikning o‘ng qismini $\Phi_0(x)$ funksiya orqali ifodalaymiz:

$$P_n(m_1 \le m \le m_2) = \frac{1}{\sqrt{2\pi}} \int_{x_1}^{x_2} e^{-x/2} dx =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{x_1}^{x_2} e^{-t/2} dt = \frac{1}{\sqrt{2\pi}} \int_{x_1}^{0} e^{-x/2} dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{x_2} e^{-x/2} dx = \Phi_0(x_2) - \Phi_0(x_1). \quad (1.14.7)$$

$\Phi_0(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2} dt$ -Laplasning funksiyasi bilan bir qatorda Gauss funksiyasi deb nomlanuvchi funksidan ham foydalaniladi:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt. \quad (1.14.8)$$

Bu funksiya uchun $\Phi(-x) + \Phi(x) = 1$ tenglik o‘rinli va u $\Phi_0(x)$ funksiya bilan

$$\Phi(x) = 0.5 + \Phi_0(x) \quad (1.14.9)$$

formula orqali bog‘langan.

1.16-misol. Sex ishlab chiqargan mahsulotining o‘rtacha 96% i sifatli. Bazada mahsulotni qabul qilib oluvchi sexning 200 ta mahsulotini tavakkaliga tekshiradi. Agar tekshirilgan mahsulotlardan sifatsizlari soni 10 tadan ko‘p bo‘lsa butun mahsulotlar partiyasi sifatsiz deb, sexga qaytariladi. Mahsulotlar partiyasining qabul qilinishi ehtimolligini toping.
Bu yerda n=200, p=0.04(mahsulotning sifatsiz bo‘lish ehtimolligi), q=0.96, m_1=0, m_2=10 va mahsulotlar partiyasining qabul qilinishi ehtimolligi $P_{200}(0 \le m \le 10)$ ni (1.14.7) formula orqali hisoblaymiz:

$$x_1 = \frac{0 - 200 \cdot 0.04}{\sqrt{200 \cdot 0.04 \cdot 0.96}} \approx -2.89, \quad x_2 = \frac{10 - 200 \cdot 0.04}{\sqrt{200 \cdot 0.04 \cdot 0.96}} \approx 0.72,$$

$$P_{200}(0 \le m \le 10) = \Phi_0(0.72) - \Phi_0(-2.89) = 0.26424 + 0.49807 = 0.7623.$$

Agar $\Phi(x)$ funksiyadan foydalansak,

$$P_{200}(0 \le m \le 10) = \Phi(0.72) - \Phi(-2.89) =$$
$$= 0.7642 - (1 - \Phi(2.89)) = 0.7642 - (1 - 0.998074) = 0.7623.$$

Laplas funksiyasi yordamida n ta bog‘liqsiz tajribada nisbiy chastotaning ehtimollikdan chetlashishi ehtimolligini hisoblash mumkin.

✓ Biror $\varepsilon > 0$ son uchun

$$P\left\{\left|\frac{n_A}{n}-p\right|\le\varepsilon\right\}=2\Phi_0\left(\varepsilon\cdot\sqrt{\frac{n}{pq}}\right) \qquad (1.14.10)$$

tenglik o‘rinli.

Haqiqatan ham, buni isbotlash uchun $\left|\frac{n_A}{n}-p\right|\le\varepsilon$ tengsizlik ehtimolligini hisoblash kerak. Buning uchun bu tengsizlikni unga teng kuchli $-\varepsilon\le\frac{n_A}{n}-p\le\varepsilon$ yoki $-\varepsilon\le\frac{n_A-np}{n}\le\varepsilon$ tengsizliklar bilan almashtiramiz. Bu tengsizliklarni musbat $\sqrt{\frac{n}{pq}}$ songa ko‘paytiramiz:

$$-\varepsilon\sqrt{\frac{n}{pq}}\le\frac{n_A-np}{\sqrt{npq}}\le\varepsilon\sqrt{\frac{n}{pq}}.$$

Agar $m=\frac{n_A-np}{\sqrt{npq}}$ belgilashni kiritsak, u holda (1.14.5) formulaga asosan:

$$P_n(-\varepsilon\sqrt{\frac{n}{pq}}\le m\le\varepsilon\sqrt{\frac{n}{pq}})\approx\frac{1}{\sqrt{2\pi}}\int_{-\varepsilon\sqrt{\frac{n}{pq}}}^{\varepsilon\sqrt{\frac{n}{pq}}}e^{-t^2/2}dt=\frac{2}{\sqrt{2\pi}}\int_0^{\varepsilon\sqrt{\frac{n}{pq}}}e^{-t^2/2}dt=2\Phi_0\left(\varepsilon\sqrt{\frac{n}{pq}}\right).$$

■

1.17-misol. Detalning nostandart bo‘lishi ehtimolligi 0.6 ga teng. n=1200 ta detal ichida nostandart detallar bo‘lishi nisbiy chastotasining p=0.6 ehtimollikdan chetlashishi absolut qiymati $\varepsilon=0.05$ dan katta bo‘lmasligi ehtimolligini toping.

(1.4.10) ga asosan,

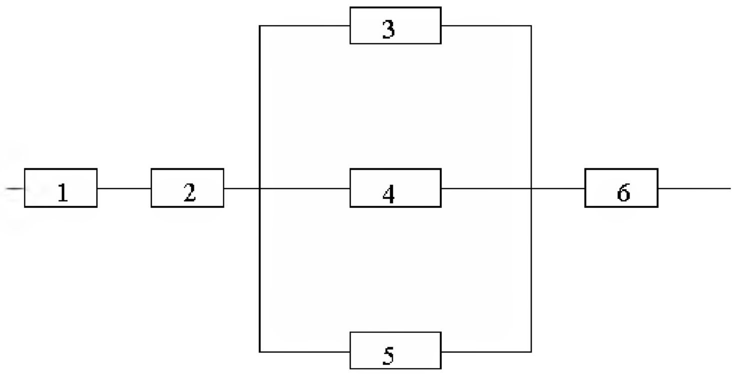
$$P_{1200}\left\{\left|\frac{n_A}{n}-0.6\right|\le0.05\right\}=2\Phi_0\left(0.05\sqrt{\frac{1200}{0.6\cdot0.4}}\right)=2\Phi_0(3.54)\approx0.9996.$$

I bobga doir misollar

1. A,B va C hodisalar uchun quyidagilarni isbotlang: a) $B=A\cdot B+\overline{A}\cdot B$;

b) $(A+B)\cdot(B+C)=A\cdot B+C$; c) $\overline{A+B}=\overline{A}\cdot\overline{B}$.

2. 11-rasmda 6 elementdan iborat sxema berilgan. A_i $(i=\overline{1,6})$ hodisalar ma’lum T vaqt oralig‘ida mos elementlarning beto‘xtov ishlashi bo‘lsa, bu hodisalar orqali ma’lum T vaqt oralig‘ida sxemaning beto‘xtov ishlashini ifodalang.



11-rasm.

3. Ixtiyoriy ikki qo‘shni raqamlari har xil bo‘lgan nechta to‘rt xonali son hosil qilish mumkin?

4. Musobaqaning 10 ta ishtirokchisiga 3 ta yutuqni necha xil usul bilan taqsimlash mumkin.

5. Ma’lum uchta kitob yonma-yon turadigan qilib, 7 ta kitobni tokchaga necha xil usul bilan taxlash mumkin.

6. Birinchi talabada 7 xil, ikkinchisida 16 xildagi kitoblar bor bo‘lsa, kitobga kitobni necha xil usul bilan almashtirishlari mumkin. 2 ta kitobga 2 ta kitobnichi?

7. 3,3,5,5,8 raqamlaridan nechta besh xonali son hosil qilish mumkin.

8. 9 qavatli bino liftiga 4 kishi kirdi. Ularning har biri bir-biriga bo‘gliqsiz ravishda ixtiyoriy qavatda chiqishlari mumkin. Ular : a) turli qavatlarda; b) bitta qavatda: c) 5-qavatda chiqishlari ehtimolliklarini toping.

9. Imtihon biletlariga kiruvchi 60 savoldan talaba 50 tasini biladi. Tavakkaliga tanlangan 3 ta savoldan: a) hammasini; b) ikkitasini bilishi ehtimolligini toping.

10. Idishda 5 ta ko‘k, 4 ta qizil va 3 ta yashil shar bor. Tavakkaliga olingan 3 ta sharning: a) bir xil rangda; b) har xil rangda; c) 2 tasi ko‘k va 1 tasi yashil rangda bo‘lishi ehtimolligini hisoblang.

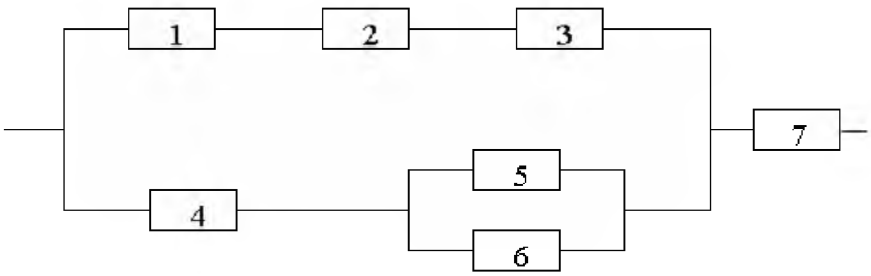
11. R radiusli doiraga teng tomonli uchburchak ichki chizilgan. Doiraga tavakkaliga tashlangan nuqtaning uchburchakka tushishi ehtimolligini toping.

12. [0,5] kesmadan tavakkaliga bitta nuqta tanlanadi. Shu nuqtadan kesmaning o‘ng oxirigacha bo‘lgan masofa 1.6 birlikdan oshmasligi ehtimolligini toping.

13. Idishda 4 ta oq, 3 ta ko‘k va 2 ta qora shar bor. Tavakkaliga, ketma-ket, bittadan 3 ta shar olindi. Birinchi shar oq, ikkinchisi ko‘k va uchinchisi qora rangda bo‘lishi ehtimolligini toping.

14. Shoshqol toshni tashlash tajribasida A={juft raqam tushishi} va B={3 dan katta raqam tushishi} hodisalari bo‘lsin. A va B hodisalar bog‘liqsizmi?

15. Quyida berilgan bir-biriga bog‘liqsiz ravishda ishlaydigan elementlardan iborat sxemaning safdan chiqishi ehtimolligini toping, $i(i=1,2,\ldots,7)$-elementning safdan chiqishi ehtimolligi 0.2 ga teng .



12-rasm.

16. Asbob ikki mikrosxemadan iborat. Birinchi mikrosxemaning 10 yil ichida ishdan chiqishi ehtimolligi 0.07, ikkinchisiniki-0.10. Bitta mikrosxema ishdan chiqgani ma’lum bo‘lsa, bu mikrosxema birinchisi ekanligi ehtimolligini toping.

17. Talaba imtihon 40 ta biletlarining faqat 30 tasiga javob bera oladi. Talabaga imtihonga birinchi bo‘lib kirishi foydalimi, yoki ikkinchi?

18. Zavod ishlab chiqargan mahsulotning 90% i sifat talablariga javob beradi. Tekshruvchi mahsulotni 0.96 ehtimollik bilan sifatli, 0.06 ehtimollik bilan sifatsiz deb topadi. Tavakkaliga olingan mahsulotning sifatli deb topilishi ehtimolligini toping.

19. Oilada 3 ta farzand bor. Agar o‘g‘il bola tug‘ilishi ehtimolligi 0.51, qiz bola tug‘lishi ehtimolligi 0.49 ga teng bo‘lsa, a) bolalarning hammasi o‘g‘illar, b) 1 tasi o‘g‘il va 2 tasi qiz bo‘lishi ehtimolliklarini hisoblang.

20. Shoshqol tosh 10 marta tashlanganda:

a) 6 raqami bir marta tushishi ehtimolligini;

b) 6 raqami kamida bir marta tushish ehtimolligini;

c) 6 raqami tushishi soni ehtimolligi maksimal qiymatga erishadigan miqdorni toping.

21. “Ehtimollar nazariyasi” fanidan ma’ruza darsida 84 ta talaba ishtirok etmoqda. Shu talabalarning ikkitasini tug‘ilgan kuni shu kuni bo‘lishi ehtimolligini toping.

22. Mahsulotning sifatsiz bo‘lishi ehtimolligi 0.02 ga teng. 200 ta mahsulotning ichida sifatsizlari bittadan ko‘p bo‘lmasligi ehtimolligiti toping.

23. A hodisaning ro‘y berish ehtimolligi 0.6 ga teng. 100 ta bog‘liqsiz tajribada A hodisaning 70 marta ro‘y berishi ehtimolligini toping.

24. Shunday m sonini topingki, 0.95 ehtimollik bilan 800 ta yangi tug‘ilgan chaqaloqlardan kamida m tasi qizlar deb aytish mumkin bo‘lsin. Qiz bola tug‘ilishi ehtimolligini 0.485 deb hisoblang.

25. Detalning nostandart bo‘lishi ehtimolligi 0.1 ga teng. Tavakkaliga olingan 400 ta detal ichida nostandart detallar bo‘lishi nisbiy chastotasining p=0.1 ehtimollikdan chetlashishi absolut qiymati $\varepsilon = 0.03$ dan katta bo‘lmasligi ehtimolligini toping.

II bob Tasodifiy moqdorlar

2.1 Tasodifiy miqdor tushunchasi

Ehtimollar nazariyasining muhim tusunchalaridan biri tasodifiy miqdor tushunchasidir.

✓ Tajriba natijasida u yoki bu qiymatni qabul qilishi oldindan ma'lum bo'lmagan miqdor *tasodifiy miqdor* deyiladi.

Tasodifiy miqdorlar lotin alifbosining bosh harflari $X,Y,Z,\ldots$(yoki grek alifbosining kichik harflari ξ(ksi), η(eta), ζ(dzeta),…) bilan qabul qiladigan qiymatlari esa kichik harflar $x_1,x_2,\ldots,y_1,y_2,\ldots,z_1,z_2,\ldots$ bilan belgilanadi.

Tasodifiy miqdorlarga misollar keltiramiz: 1) X-tavakkaliga olingan mahsulotlar ichida sifatsizlari soni; 2) Y-n ta o'q uzilganda nishonga tekkanlari soni; 3) Z-asbobning beto'htov ishlash vaqti; 4) U-[0,1] kesmadan tavakkaliga tanlangan nuqtaning koordinatalari; 5) V-bir kunda tug'iladigan chaqaloqlar soni va h.k..

✓ Agar tasodifiy miqdor(t.m.) chekli yoki sanoqli qiymatlar qabul qilsa, bunday t.m. *diskret tipdagi t.m.* deyiladi.

✓ Agar t.m. qabul qiladigan qiymatlari biror oraliqdan iborat bo'lsa *uzluksiz tipdagi t.m.* deyiladi.

Demak, diskret t.m. bir-biridan farqli alohida qiymatlarni, uzluksiz t.m. esa biror oraliqdagi ihtiyoriy qiymatlarni qabul qilar ekan. Yuqoridagi X va Y t.m.lar diskret, Z esa uzluksiz t.m. bo'ladi.

Endi t.m.ni qat'iy ta'rifini keltiramiz.

✓ Ω elementar hodisalar fazosida aniqlangan X sonli funksiya t.m. deyiladi, agar har bir ω elementar hodisaga $X(\omega)$ conni mos qo'ysa, yani $X=X(\omega)$, $\omega\in\Omega$.

Masalan, tajriba tangani 2 marta tashlashdan iborat bo'lsin. Elementar hodisalar fazosi $\Omega=\{\omega_1,\omega_2,\omega_3,\omega_4\}$, $\omega_1=GG, \omega_2=GR, \omega_3=RG, \omega_4=RR$ bo'ladi. X-gerb chiqishlari soni bo'lsin, u holda X t.m. qabul qiladigan qiymatlari: $X(\omega_1)=2$, $X(\omega_2)=1$, $X(\omega_3)=1$, $X(\omega_4)=0$.

Agar Ω chekli yoki sanoqli bo'lsa, u holda Ω da aniqlangan ixtiyoriy funksiya t.m. bo'ladi. Umuman, $X(\omega)$ funksiya shunday bo'lishi kerakki: $\forall x\in R$ da $A=\{\omega: X(\omega)<x\}$ hodisa S σ-algebrasiga tegishli bo'lishi kerak.

2.2 Diskret tasodifiy miqdorning taqsimot qonuni

X-diskret t.m. bo‘lsin. *X* t.m. $x_1, x_2, ..., x_n, ...$ qiymatlarni mos $p_1, p_2, ..., p_n, ...$ ehtimolliklar bilan qabul qilsin:

X	x_1	x_2	$\cdots$	x_n	$\cdots$
P	p_1	p_2	$\cdots$	p_n	$\cdots$

jadval diskret t.m. taqsimot qonuni jadvali deyiladi. Diskret t.m. taqsimot qonunini $p_i = P\{X = x_i\}, i = 1, 2, ..., n, ...$ ko‘rinishda yozish ham qulay.

$\{X = x_1\}, \{X = x_2\}, ...$ hodisalar birgalikda bo‘lmaganligi uchun ular to‘la gruppani tashkil etadi va ularning ehtimolliklari yig‘indisi birga teng bo‘ladi, ya’ni $\sum_i p_i = \sum_i P\{X = x_i\} = 1$.

✓ *X* t.m. *diskret t.m.* deyiladi, agar $x_1, x_2, ...$ chekli yoki sanoqli to‘plam bo‘lib, $P\{X = x_i\} = p_i > 0 \ (i = 1, 2, ...)$ va $p_1 + p_2 + ... = 1$ tenglik o‘rinli bo‘lsa.

✓ *X* va *Y* diskret t.m.lar *bog‘liqsiz* deyiladi, agar $A_i = \{X = x_i\}$ va $B_i = \{Y = y_j\}$ hodisalar $\forall i = 1, 2, ..., n, \ j = 1, 2, ..., m$ da bog‘liqsiz bo‘lsa, ya’ni $P\{X = x_i, Y = y_j\} = P\{X = x_i\} \cdot P\{Y = y_j\}, \ n, m \geq \infty$.

2.1-misol. 10 ta lotoreya biletida 2 tasi yutuqli bo‘lsa, tavakkaliga olingan 3 ta lotoreya biletlari ichida yutuqlilari soni X t.m.ning taqsimot qonunini toping.

X t.m.ni qabul qilishi mumkin bo‘lgan qiymatlari $x_1 = 0, x_2 = 1, x_3 = 2$. Bu qiymatlarning mos ehtimolliklari esa

$$p_1 = P\{X = 0\} = \frac{C_2^0 \cdot C_8^3}{C_{10}^3} = \frac{56}{120} = \frac{7}{15};$$

$$p_2 = P\{X = 1\} = \frac{C_2^1 \cdot C_8^2}{C_{10}^3} = \frac{56}{120} = \frac{7}{15};$$

$$p_3 = P\{X = 2\} = \frac{C_2^2 \cdot C_8^1}{C_{10}^3} = \frac{8}{120} = \frac{1}{15}.$$

X t.m. taqsimot qonunini jadval ko‘rinishida yozamiz:

X	0	1	2
P	$\frac{7}{15}$	$\frac{7}{15}$	$\frac{1}{15}$

$$\sum_{i=1}^{3} p_i = \frac{7}{15} + \frac{7}{15} + \frac{1}{15} = 1$$

2.3 Taqsimot funksiyasi va uning xossalari

Diskret va uzluksiz t.m.lar taqsimotlarini berishning universal usuli ularning taqsimot funksiyalarini berishdir. Taqsimot funksiya $F(x)$ orqali belgilanadi.

✓ $F(x)$ funksiya X t.m.ning *taqsimot funksiyasi* $\forall x \in R$ son uchun quyidagicha aniqlanadi:

$$F(x) = P\{X < x\} = P\{\omega : X(\omega) < x\}. \qquad (2.3.1)$$

Taqsimot funksiyasi quyidagi xossalarga ega:

1. $F(x)$ chegaralangan:
$$0 \le F(x) \le 1.$$

2. $F(x)$ kamaymaydigan funksiya: agar $x_1 < x_2$ bo'lsa, u holda $F(x_1) \le F(x_2)$.

3. $F(-\infty) = \lim\limits_{x \to -\infty} F(x) = 0, \quad F(+\infty) = \lim\limits_{x \to +\infty} F(x) = 1$.

4. $F(x)$ funksiya chapdan uzluksiz:
$$\lim_{x \to x_0 - 0} F(x) = F(x_0).$$

Isboti: 1. Bu xossa (2.3.1) va ehtimollikning xossalaridan kelib chiqadi.

2. $A = \{X < x_1\}, B = \{X < x_2\}$ hodisalarni kiritamiz. Agar $x_1 < x_2$ bo'lsa, u holda $A \subseteq B$ va $P(A) \le P(B)$, ya'ni $P(X < x_1) \le P(X < x_2)$ yoki $F(x_1) \le F(x_2)$.

3. $\{X < -\infty\} = \varnothing$ va $\{X < +\infty\} = \Omega$ ekanligi va ehtimollikning xossasiga ko'ra

$$F(-\infty) = P\{X < -\infty\} = P\{\varnothing\} = 0$$
$$F(+\infty) = P\{X < +\infty\} = P\{\Omega\} = 1.$$

4. $A = \{X < x_0\}, A_n = \{X < x_n\}$ hodisalarni kiritamiz. Bu yerda $\{x_n\}$ ketma-ketlik monoton o'suvchi, $x_n \uparrow x_0$. A_n hodisalar ketma-ketligi ham o'suvchi bo'lib, $\bigcup_n A_n = A$. U holda $P(A_n) \to P(A)$, ya'ni $\lim\limits_{x \uparrow x_0} F(x) = F(x_0)$. ■

Diskret t.m. taqsimot funksiyasi quyidagicha ifodalanadi:

$$F(x) = \sum_{x_i < x} p_i \,. \qquad (2.3.2)$$

2.2-misol. 2.1-misoldagi X t.m. taqsimot funksiyasini topamiz.

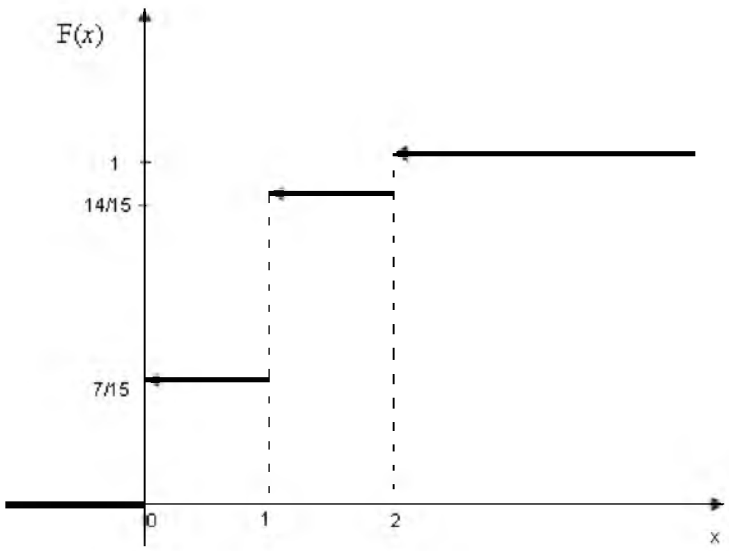
X	0	1	2
P	$\frac{7}{15}$	$\frac{7}{15}$	$\frac{1}{15}$

1. Agar $x \le 0$ bo‘lsa, $F(x) = P\{X < 0\} = 0$;
2. Agar $0 < x \le 1$ bo‘lsa, $F(x) = P\{X < 1\} = P\{X = 0\} = \frac{7}{15}$;
3. Agar $1 < x \le 2$ bo‘lsa, $F(x) = P\{X = 0\} + P\{X = 1\} = \frac{7}{15} + \frac{7}{15} = \frac{14}{15}$;
4. Agar $x > 2$ bo‘lsa, $F(x) = P\{X = 0\} + P\{X = 1\} + P\{X = 2\} = \frac{7}{15} + \frac{7}{15} + \frac{1}{15} = 1$.

Demak,

$$F(x) = \begin{cases} 0, \; agar \; x \le 0 \\ 7/15, \; agar \; 0 < x \le 1 \\ 14/15, \; agar \; 1 < x \le 2 \\ 1, \; agar \; x > 2 \end{cases}$$

$F(x)$ taqsimot funksiya grafigi 13-rasmda keltirilgan.



13-rasm.

✓ X t.m. uzluksiz deyiladi, agar uning taqsimot funksiyasi ixtiyoriy nuqtada uzluksiz bo‘lsa.

Agar $F(x)$ taqsimot funksiya uzluksiz t.m. taqsimot funksiyasi bo‘lsa, taqsimot funksiyaning 1-4 xossalaridan quyidagi natijalarni keltirish mimkin:

1. X t.m.ning [a,b) oraliqda yotuvchi qiymatni qabul qilish ehtimolligi taqsimot funksiyaning shu oraliqdagi orttirmasiga teng:

$$P\{a \le X < b\} = F(b) - F(a). \tag{2.3.3}$$

2. X uzluksiz t.m.ning tayin bitta qiymatni qabul qilishi ehtimolligi nolga teng:

$$P\{X = x_i\} = 0$$

1-natijada $[a,b]$, $(a,b]$, (a,b) oraliqlar uchun ham (2.3.3) tenglik o‘rinli, ya’ni

$$P\{a \le X < b\} = P\{a \le X \le b\} = P\{a < X \le b\} = P\{a < X < b\} = F(b) - F(a).$$

Masalan, $P\{a \le X < b\} = P\{X = a\} + P\{a < X < b\} = P\{a < X < b\}$.

Isboti. 1. $a<b$ bo‘lgani uchun $\{X < b\} = \{X < a\} + \{a \le X < b\}$. $\{X < a\}$ va $\{a \le X < b\}$ hodisalar birgalikda bo‘lmagani uchun $P\{X < b\} = P\{X < a\} + +P\{a \le X < b\}$. $P\{a \le X < b\} = P\{X < b\} - P\{X < a\} = F(b) - F(a)$.

2. (2.3.3.) tenglikni $[a,x)$ oraliqqa tatbiq etamiz: $P\{a \le X < x\} = F(x) - F(a)$. $F(x)$ funksiya a nuqtada uzluksiz bo‘lgani uchun $\lim\limits_{x \to a} F(x) = F(a)$.

$\lim\limits_{x \to a} P\{a \le X < x\} = P\{X = a\} = \lim\limits_{x \to a} F(x) - F(a) = F(a) - F(a) = 0$. ■

2.4 Zichlik funksiyasi va uning xossalari

Uzluksiz t.m.ni asosiy xarakteristikasi zichlik funksiya hisoblanadi.

✓ Uzluksiz t.m. *zichlik funksiyasi* deb, shu t.m. taqsimot funksiyasidan olingan birinchi tartibli hosilaga aytiladi.

Uzluksiz t.m. zichlik funksiyasi $f(x)$ orqali belgilanadi. Demak,

$$f(x) = F'(x). \tag{2.4.1}$$

Zichlik funksiyasi quyidagi xossalarga ega:

1. $f(x)$ funksiya manfiy emas, ya'ni

$$f(x) \geq 0.$$

2. X uzluksiz t.m.ning $[a,b]$ oraliqqa tegishli qiymatni qabul qilishi ehtimolligi zichlik funksiyaning a dan b gacha olingan aniq integralga teng, ya'ni

$$P\{a \leq X \leq b\} = \int_a^b f(x)dx.$$

3. Uzluksiz t.m. taqsimot funksiyasi zichlik funksiya orqali quyidagicha ifodalanadi:

$$F(x) = \int_{-\infty}^{x} f(t)dt. \qquad (2.4.2)$$

4. Zichlik funksiyasidan $-\infty$ dan $+\infty$ gacha olingan xosmas integral birga tengdir

$$\int_{-\infty}^{+\infty} f(x)dx = 1.$$

Isbotlar: 1. $F(x)$ kamaymaydigan funksiya bo'lgani uchun $F'(x) \geq 0$, ya'ni $f(x) \geq 0$.

2. $P\{a \leq X < b\} = F(b) - F(a)$ tenglikdan Nyuton-Leybnis formulasiga asosan:

$$F(b) - F(a) = \int_a^b F'(x)dx = \int_a^b f(x)dx.$$

Bu yerdan $P\{a \leq X \leq b\} = \int_a^b f(x)dx$.

3. 2-xossadan foydalanamiz:

$$F(x) = P\{X < x\} = P\{-\infty < X < x\} = \int_{-\infty}^{x} f(t)dt.$$

4. Agar 2-xossada $a = -\infty$ va $b = +\infty$ deb olsak, u holda muqarrar $X \in (-\infty, +\infty)$ ga hodisaga ega bo'lamiz, u holda

$$\int_{-\infty}^{+\infty} f(x)dx = P\{-\infty < X < +\infty\} = P\{\Omega\} = 1.$$

■

2.3.-misol. X t.m. zichlik funksiyasi $f(x) = \dfrac{a}{1+x^2}$ tenglik bilan berilgan. O'zgarmas a parametrni toping.

Zichlik funksiyaning 4-xossasiga ko‘ra $\int_{-\infty}^{+\infty}\frac{a}{1+x^2}dx=1$, ya’ni $a\cdot\lim_{\substack{d\to+\infty\\ c\to-\infty}}\int_c^d\frac{1}{1+x^2}dx=a\cdot\lim_{\substack{d\to+\infty\\ c\to-\infty}} arctgx\,|_c^d=a\cdot\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)=a\cdot\pi=1$. Demak, $a=\frac{1}{\pi}$.

2.5 Tasodifiy miqdorning sonli xarakteristikalari

X diskret t.m. taqsimot qonuni berilgan bo‘lsin: $\{\,p_i=P\{X=x_i\},\,i=1,2,...,n,...\,\}$.

Matematik kutilma

✓ X t.m. *matematik kutilmasi* deb, $\sum_{i=1}^{\infty}x_ip_i$ qator yig‘indisiga aytiladi va

$$MX=\sum_{i=1}^{\infty}x_ip_i \qquad (2.5.1)$$

orqali belgilanadi.

Matematik kutilmaning ma’nosi shuki, u t.m. o‘rta qiymatini ifodalaydi. Haqiqatan ham $\sum_{i=1}^{\infty}p_i=1$ ekanligini hisobga olsak, u holda

$$MX=\sum_{i=1}^{\infty}x_ip_i=\frac{\sum_{i=1}^{\infty}x_ip_i}{\sum_{i=1}^{\infty}p_i}=x_{o'rtacha}.$$

✓ Uzluksiz t.m. *matematik kutilmasi* deb

$$MX=\int_{-\infty}^{+\infty}x\cdot f(x)dx \qquad (2.5.2)$$

integralga aytiladi. (2.5.2) integral absolut yaqinlashuvchi, ya’ni $\int_{-\infty}^{+\infty}|x|\cdot f(x)dx<\infty$ bo‘lsa matematik kutilma chekli, aks holda matematik kutilma mavjud emas deyiladi.

Matematik kutilmaning xossalari:

1. O‘zgarmas sonning matematik kutilmasi shu sonning o‘ziga teng, ya’ni

$$MC=C.$$

2. O‘zgarmas ko‘paytuvchini matematik kutilish belgisidan tashqariga chiqarish mumkin,

$$M(CX)=CMX.$$

3. Yig‘indining matematik kutilmasi matematik kutilmalar yig‘indisiga teng,

$$M(X+Y)=MX+MY.$$

4. Agar X⊥Y bo‘lsa,

$$M(X\cdot Y)=MX\cdot MY.$$

Isbotlar: 1. O‘zgarmas C sonni faqat 1 ta qiymatni bir ehtimollik bilan qabul qiluvchi t.m. sifatida qarash mumkin. Shuning uchun $MC=C\cdot P\{X=C\}=C\cdot 1=C$.

2. $C\cdot X$ diskret t.m. $C\cdot x_i\ (i=\overline{1,n})$ qiymatlarni p_i ehtimolliklar bilan qabul qilsin, u holda $MCX=\sum_{i=1}^{n} C\cdot x_i p_i = C\sum_{i=1}^{n} x_i p_i = C\cdot MX$.

3. $X+Y$ diskret t.m. x_i+y_i qiymatlarni $p_{ij}=P\{X=x_i, Y=y_j\}$ ehtimolliklar bilan qabul qiladi, u holda ixtiyoriy n va m lar uchun

$$M(X+Y)=\sum_{i=1}^{n}\sum_{j=1}^{m}(x_i+y_j)p_{ij}=\sum_{i=1}^{n}\sum_{j=1}^{m}x_i p_{ij}+\sum_{i=1}^{n}\sum_{j=1}^{m}y_j p_{ij}=$$

$$=\sum_{i=1}^{n}x_i\sum_{j=1}^{m}p_{ij}+\sum_{j=1}^{m}y_i\sum_{i=1}^{n}p_{ij}=\sum_{i=1}^{n}x_i p_i+\sum_{j=1}^{m}y_j p_j=MX+MY$$

Bu yerda $\sum_{j=1}^{m} p_{ij}=p_i$ va $\sum_{i=1}^{n} p_{ij}=p_j$ bo‘ladi. Chunki,

$$\bigcup_{j=1}^{m}\{X=x_i;Y=y_j\}=\{X=x_i\}\bigcup_{j=1}^{m}\{Y=y_j\}=\{X=x_i\}\bigcap\Omega=\{X=x_i\},$$

$$p_i=P\{X=x_i\}=P\left(\bigcup_{j=1}^{m}\{X=x_i;Y=y_j\}\right)=\sum_{j=1}^{m}P\{X=x_i;Y=y_j\}=\sum_{j=1}^{m}p_{ij}.$$

4. Agar $X\perp Y$ bo‘lsa, u holda

$$p_{ij}=P\{X=x_i,Y=y_j\}=P\{X=x_i\}\cdot P\{Y=y_j\}=p_i\cdot p_j \text{ va}$$

$$MXY=\sum_{i=1}^{n}\sum_{j=1}^{m}x_iy_i\underbrace{P\{X=x_i,Y=y_j\}}_{p_{ij}}=$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{m}x_iy_i\underbrace{P\{X=x_i\}}_{p_i}\underbrace{P\{Y=y_j\}}_{p_j}=\sum_{i=1}^{n}x_ip_i\sum_{j=1}^{m}y_ip_j=MX\cdot MY.$$

■

Matematik kutilmaning xossalari t.m. uzluksiz bo'lganda ham hiddi shunga o'xshash isbotlanadi. Masalan, $MCX=\int_{-\infty}^{+\infty}C\cdot x\cdot f(x)dx=C\int_{-\infty}^{+\infty}x\cdot f(x)dx=C\cdot MX$.

2.4.-misol. X diskret t.m. taqsimot qonuni berilgan bo'lsa, X t.m.ning matematik kutilmasini toping.

X	500	50	10	1	0
P	0.01	0.05	0.1	0.15	0.69

MX=500·0.01+50·0.05+10·0.1+1·0.15+0·0.69=8.65.

2.5.-misol. X uzluksiz t.m. zichlik funksiyasi berilgan

$$f(x)=\begin{cases}0, & x\notin(0,1)\\ C\cdot x^2, & x\in(0,1)\end{cases}.$$

C va MX ni toping.

Zichlik funksiyaning 4-xossasiga ko'ra $\int_{-\infty}^{+\infty}f(x)dx=1$. Demak, $C\int_0^1 x^2dx=C\cdot\frac{x^3}{3}\Big|_0^1=C\cdot\frac{1}{3}=1,\quad C=3$ va $f(x)=\begin{cases}0, & x\notin(0,1)\\ 3x^2, & x\in(0,1)\end{cases}$.

Endi matematik kutilmani hisoblaymiz:

$$MX=\int_{-\infty}^{+\infty}x\cdot f(x)dx=3\int_0^1 x\cdot x^2dx=\frac{3}{4}.$$

Dispersiya

✓ X t.m. dispersiyasi deb, $M(X-MX)^2$ ifodaga aytiladi.

Dispersiya DX orqali belgilanadi. Demak,

$$DX=M(X-MX)^2. \qquad (2.5.3)$$

Agar X dickret t.m. bo‘lsa,

$$DX=\sum_{i=1}^{\infty}(x_i-MX)^2\cdot p_i\,, \qquad (2.5.4)$$

Agar X uzluksiz t.m. bo‘lsa,

$$DX=\int_{-\infty}^{+\infty}(x-MX)^2\cdot f(x)dx \qquad (2.5.5)$$

T.m. dispersiyasini hisoblash uchun quyidagi formula qulaydir:

$$DX=MX^2-(MX)^2 \qquad (2.5.6)$$

Bu formula matematik kutilma xossalari asosida quyidagicha keltirib chiqariladi:

$$DX=M(X-MX)^2=M(X^2-2XMX+(MX)^2)=MX^2-M(2XMX)+M(MX)^2=$$
$$=MX^2-2MXMX+(MX)^2=MX^2-(MX)^2$$

Dispersiyaning xossalari:

1. O‘zgarmas sonning dispersiyasi nolga teng $DC=0$.
2. O‘zgarmas ko‘paytuvchini kvadratga ko‘tarib, dispersiya belgisidan tashqariga chiqarish mumkin,

$$D(CX)=C^2DX.$$

3. Agar $X\perp Y$ bo‘lsa,

$$D(X+Y)=DX+DY.$$

Isbotlar: 1. $DC=M(C-MC)^2=M(C-C)^2=M0=0$.

2. $D(CX)=M(CX-M(CX))^2=M(CX-CMX)^2=M(C^2(X-MX)^2)=$
$=C^2M(X-MX)^2=C^2DX$.

3. (2.5.6.) formulaga ko‘ra

$$D(X+Y)=M(X+Y)^2-(M(X+Y))^2=MX^2+2MXY+MY^2-(MX)^2-2MXMY-(MY)^2=$$
$$=MX^2-(MX)^2+MY^2-(MY)^2+2(MXY-MXMY)=DX+DY+2(MXMY-MXMY)=DX+DY$$

■

2.6.-misol. X diskret t.m. taqsimot qonuni berilgan:

X	-1	0	1	2
P	0.2	0.1	0.3	0.4

MX va DX ni hisoblaymiz:

$MX=-1\cdot0.2+0\cdot0.1+1\cdot0.3+2\cdot0.4=0.9$,

$DX=(-1)^2\cdot0.2+1^2\cdot0.3+2^2\cdot0.4-(0.9)^2=1.29$.

✓ X t.m. *o‘rtacha kvadratik tarqoqligi*(*chetlashishi*) deb, dispersiyadan olingan kvadrat ildizga aytiladi:

$$\sigma_X = \sqrt{DX} \qquad (2.5.7)$$

Dispersiyaning xossalaridan o‘rtacha kvadratik tarqoqlikning xossalari kelib chiqadi: 1. $\sigma_C = 0$; 2. $\sigma_{CX} = |C|\sigma_X$;

2.6 Ba’zi muhim taqsimotlar
Binomial taqsimot

✓ X diskret t.m. *binomial qonun* bo‘yicha taqsimlangan deyiladi, agar u 0,1,2,…n qiymatlarni

$$p_m = P\{X = m\} = C_n^m p^m q^{n-m}, \qquad (2.6.1)$$

ehtimollik bilan qabul qilsa.
Bu yerda $0 < p < 1,\ q = 1 - p,\ m = 0,1,...,n$.

Binomial qonun bo‘yicha taqsimlangan X diskret t.m. yaqsimot qonuni quyidagi ko‘rinishga ega:

X=m	0	1	2	…	m	…	n
$p_m = P\{X = m\}$	q^n	$C_n^1 p^1 q^{n-1}$	$C_n^2 p^2 q^{n-2}$	…	$C_n^m p^m q^{n-m}$	…	p^n

Nyuton binomiga asosan $\sum_{m=0}^{n} p_m = (p+q)^n = 1$. Bunday taqsimotni $Bi(n, p)$ orqali belgilaymiz.

Uning taqsimot funksiyasi quyidagicha bo‘ladi:

$$F(x) = \begin{cases} 0, \ agar\ x \le 0 \\ \sum_{m<x} C_n^m p^m q^{n-m}, \ agar\ 0 < x \le n \\ 1, \ agar\ n < x. \end{cases}$$

Endi bu taqsimotning sonli xarakteristikalarini hisoblaymiz.

$$MX = \sum_{m=0}^{n} m \cdot P\{X = m\} = \sum_{m=1}^{n} m \cdot P\{X = m\} = \sum_{m=1}^{n} m \cdot C_n^m p^m q^{n-m} = np \sum_{m=1}^{n} C_{n-1}^{m-1} p^{m-1} q^{n-m} =$$

$$= np(p+q)^{n-1} = np.$$

$DX=\sum_{m=0}^{n} m^2 P\{X=m\}-(np)^2=\sum_{m=1}^{n} m^2 C_n^m p^m q^{n-m}-(np)^2=|\, m^2=m(m-1)+m$

almashtirish bajaramiz$|=n(n-1)p^2\sum_{m=2}^{n} C_{n-2}^{m-2} p^{m-2} q^{n-m}+np\sum_{m=1}^{n} C_{n-1}^{m-1} p^{m-1} q^{n-m}-(np)^2=$

$n(n-1)p^2+np-(np)^2=npq$.

Demak, $MX=np;\quad DX=npq$.

Puasson taqsimoti

✓ Agar X t.m. 0,1,2,…m,… qiymatlarni

$$p_m=P\{X=m\}=\frac{a^m\cdot e^{-a}}{m!} \qquad (2.6.2)$$

ehtimolliklar bilan qabul qilsa, u *Puasson qonuni* bo‘yicha taqsimlangan t.m. deyiladi. Bu yerda *a* biror musbat son.

Puasson qonuni bo‘yicha taqsimlangan X diskret t.m.ning taqsimot qonuni quyidagi ko‘rinishga ega:

X=m	0	1	2	…	m	…
$p_m=P\{X=m\}$	e^{-a}	$\frac{a\cdot e^{-a}}{1!}$	$\frac{a^2\cdot e^{-a}}{2!}$	…	$\frac{a^m\cdot e^{-a}}{m!}$	…

Teylor yoyilmasiga asosan, $\sum_{m=0}^{\infty} p_m=e^{-a}\sum_{m=0}^{\infty}\frac{a^m}{m!}=e^{-a}\cdot e^{a}=1$. Bu taqsimotni $\Pi(a)$ orqali belgilaymiz. Uning taqsimot funksiyasi quyidagicha bo‘ladi:

$$F(x)=\begin{cases}0,\ agar\ m\le 0\\ \sum_{m<x}\frac{a^m\cdot e^{-a}}{m!},\ agar\ 0<m\le x\end{cases}$$

Endi bu taqsimotning sonli xarakteristikalarini hisoblaymiz:

$$MX=\sum_{m=0}^{\infty} m\cdot\frac{a^m\cdot e^{-a}}{m!}=e^{-a}\sum_{m=1}^{\infty} m\cdot\frac{a^m}{m!}=a\cdot e^{-a}\sum_{m=1}^{\infty}\frac{a^{m-1}}{(m-1)!}=a\cdot e^{-a}\cdot e^{a}=a,$$

$$DX=\sum_{m=0}^{\infty}m^2\cdot\frac{a^m\cdot e^{-a}}{m!}-a^2=a\sum_{m=1}m\cdot\frac{a^{m-1}\cdot e^{-a}}{(m-1)!}-a^2=$$

$$=a\cdot\left[\sum_{k=0}^{\infty}k\frac{a^k\cdot e^{-a}}{k!}+\sum_{k=0}^{\infty}\frac{a^k\cdot e^{-a}}{k!}\right]-a^2=a(a+1)-a^2=a$$

Demak, $MX=a;\quad DX=a$.

Geometrik taqsimot

✓ Agar X t.m. 1,2,…m,… qiymatlarni

$$p_m=P\{X=m\}=q^{m-1}p \qquad (2.6.3)$$

ehtimolliklar bilan qabul qilsa, u *geometrik qonuni* bo‘yicha taqsimlangan t.m. deyiladi. Bu yerda $p=1-q\in(0,1)$.

Geometrik qonun bo‘yicha taqsimlangan t.m.larga misol sifatida quyidagilarni olish mumkin: sifatsiz mahsulot chiqqunga qadar tekshirilgan mahsulotlar soni; gerb tomoni tushgunga qadar tashlangan tangalar soni; nishonga tekkunga qadar otilgan o‘qlar soni va hokazo.

Geometrik qonun bo‘yicha taqsimlangan X diskret t.m. taqsimot qonuni quyidagi ko‘rinishga ega:

$X=m$	1	2	…	m	…
$p_m=P\{X=m\}$	p	qp	…	$q^m p$	…

$$\sum_{m=1}^{\infty}q^{m-1}p=p\sum_{m=1}^{\infty}q^{m-1}=p\cdot\frac{1}{1-q}=\frac{p}{p}=1,$$

chunki p_m ehtimolliklar geometrik progressiyani tashkil etadi: $p,qp,q^2p,q^3p,\ldots$. Shuning uchun ham (2.6.3) taqsimot geometrik taqsimot deyiladi va $Ge(p)$ orqali belgilanadi.

Uning taqsimot funksiyasi quyidagicha bo‘ladi:

$$F(x)=\begin{cases}0,\ agar\ m<1\\ \sum\limits_{m<x}q^{m-1}p,\ agar\ 1\le m\le x\end{cases}$$

Endi bu taqsimotning sonli xarakteristikalarini hisoblaymiz:

$$MX=\sum_{m=1}^{\infty} m\cdot q^{m-1}p = p\sum_{m=1}^{\infty} m\cdot q^{m-1} = p\left(\sum_{m=0}^{\infty} q^m\right)_q' = p\left(\frac{1}{1-q}\right)_q' =$$

$$= p\cdot\frac{1}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p},$$

$$DX=\sum_{m=1}^{\infty} m^2\cdot q^{m-1}p - \frac{1}{p^2} = (m^2 = m(m-1)+m \text{ almashtirishni bajaramiz}) =$$

$$\sum_{m=1}^{\infty} m\cdot(m-1)q^{m-1}p + \sum_{m=1}^{\infty} m\cdot q^{m-1}p - \frac{1}{p^2} = pq\sum_{m=1}^{\infty} m\cdot(m-1)q^{m-2} + \frac{1}{p} - \frac{1}{p^2} =$$

$$q\left(\sum_{m=0}^{\infty} q^m\right)_{q^2}'' + \frac{1}{p} - \frac{1}{p^2} = pq\frac{2}{(1-q)^3} + \frac{1}{p} - \frac{1}{p^2} = \frac{2pq}{p^3} + \frac{1}{p} - \frac{1}{p^2} = \frac{q}{p^2}.$$

Demak, $MX=\frac{1}{p}$; $DX=\frac{q}{p^2}$.

Tekis taqsimot

✓ Agar uzluksiz X t.m. zichlik funksiyasi

$$f(x)=\begin{cases}\frac{1}{b-a}, \text{ agar } x\in[a,b],\\ 0, \text{ agar } x\notin[a,b]\end{cases} \qquad (2.6.4)$$

ko‘rinishda berilgan bo‘lsa, u *[a,b] oraliqda tekis taqsimlangan* t.m. deyiladi.

Bu t.m.ning grafigi 14-rasmda berilgan. $[a,b]$ oraliqda tekis taqsimlangan X t.m. ni $X\sim R[a,b]$ ko‘rinishda belgilanadi. $X\sim R[a,b]$ uchun taqsimot funksiyasini topamiz. (2.4.2) formulaga ko‘ra agar $a\le x\le b$ bo‘lsa

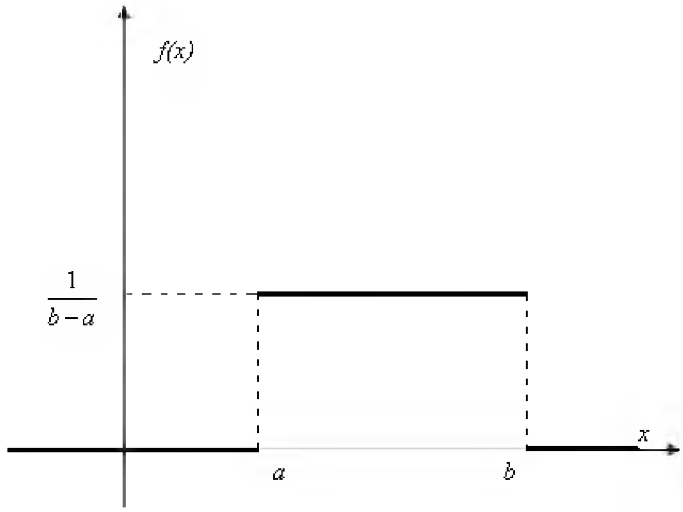
$$F(x)=\int_a^x\frac{dt}{b-a} = \left.\frac{t}{b-a}\right|_a^x = \frac{x-a}{b-a},$$

agar $x<a$ bo‘lsa, $F(x)=0$ va $x>b$ bo‘lsa,

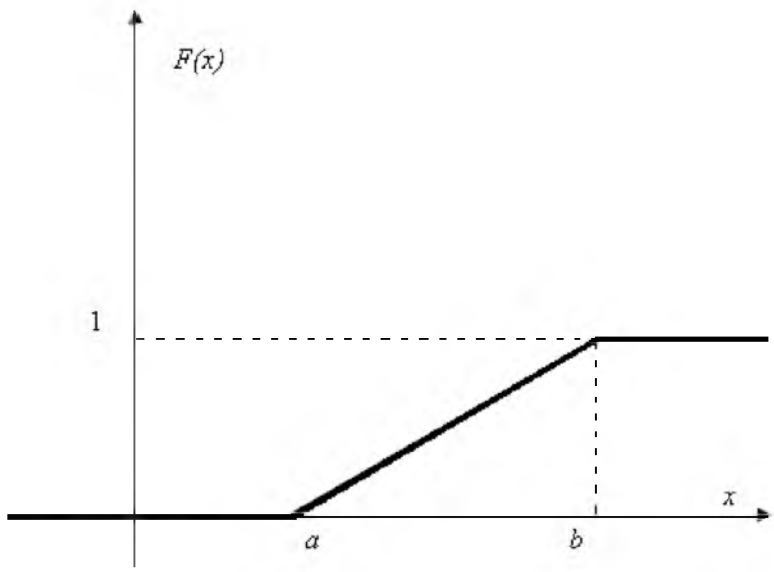
$$F(x)=\int_{-\infty}^a 0dt + \int_a^b\frac{dt}{b-a} + \int_b^x 0dt = \left.\frac{t}{b-a}\right|_a^b = 1 \text{ bo‘ladi. Demak,}$$

$$F(x)=\begin{cases}0, & \text{agar } x<a \text{ bo'lsa,}\\ \dfrac{x-a}{b-a}, & \text{agar } a\le x\le b \text{ bo'lsa,}\\ 1, & \text{agar } b<x \text{ bo'lsa,}\end{cases}$$

$F(x)$ taqsimot funksiyaning grafigi 15-rasmda keltirilgan.



14-rasm.



15-rasm.

$X \sim R[a,b]$ t.m. uchun MX va DX larni hisoblaymiz:

$$MX=\int_{-\infty}^{a}x\cdot 0dx+\int_{a}^{b}\frac{x}{b-a}dx+\int_{b}^{+\infty}x\cdot 0dx=\left.\frac{x^2}{2(b-a)}\right|_a^b=\frac{b^2-a^2}{2(b-a)}=\frac{a+b}{2},;$$

$$DX=\int_a^b\left(x-\frac{a+b}{2}\right)^2\cdot\frac{dx}{b-a}=\frac{1}{b-a}\cdot\frac{1}{3}\left.\left(x-\frac{a+b}{2}\right)^3\right|_a^b=$$

$$=\frac{1}{3(b-a)}\left(\frac{(b-a)^3}{8}-\frac{(a-b)^3}{8}\right)=\frac{(b-a)^2}{12}.$$

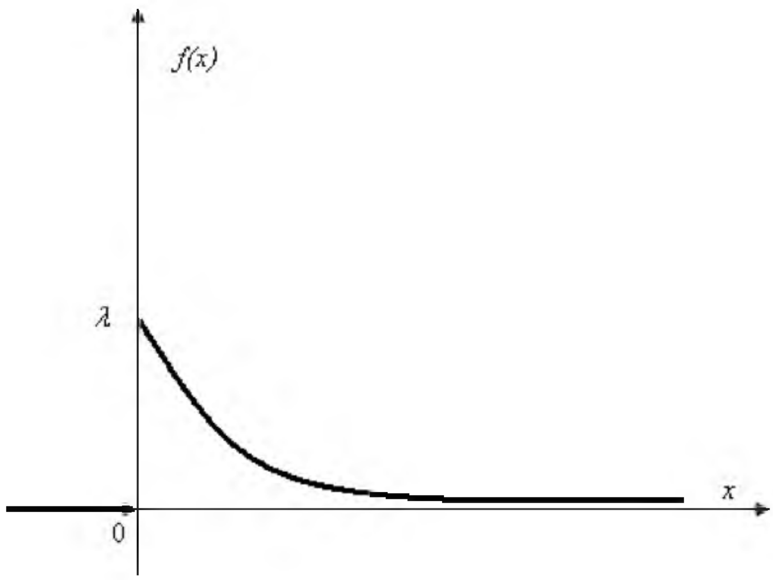
Demak, $MX=\dfrac{a+b}{2}$, $DX=\dfrac{(b-a)^2}{12}$.

Ko‘rsatkichli taqsimot

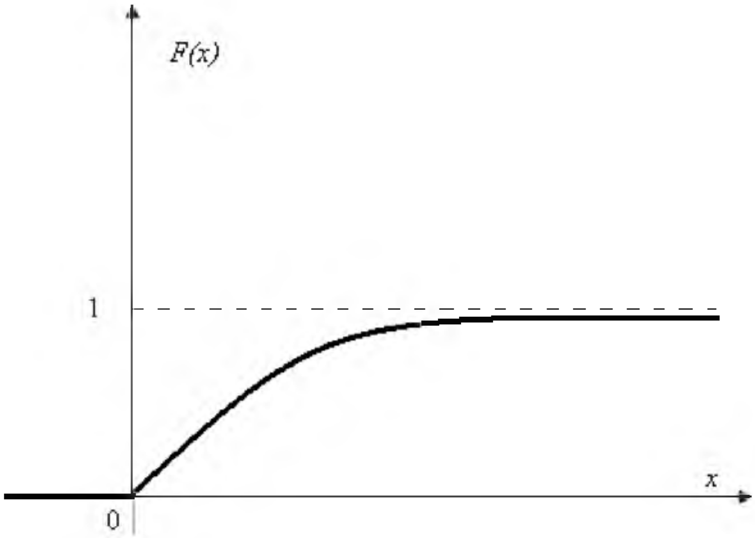
✓ Agar uzluksiz X t.m. zichlik funksiyasi

$$f(x)=\begin{cases}\lambda e^{-\lambda x}, \text{ agar } x\geq 0,\\ 0, \text{ agar } x<0\end{cases} \qquad (2.6.5)$$

ko‘rinishda berilgan bo‘lsa, X t.m. *ko‘rsatkichli qonun* bo‘yicha taqsimlangan t.m. deyiladi. Bu yerda λ biror musbat son. λ parametrli ko‘rsatkichli taqsimot $E(\lambda)$ orqali belgilanadi. Uning grafigi 16-rasmda keltirilgan.



16-rasm.



17-rasm.

Taqsimot funksiyasi quyidagicha ko‘rinishga ega bo‘ladi:

$$F(x)=\begin{cases}1-e^{-\lambda x}, \text{ agar } x\geq 0,\\ 0, \text{ agar } x<0.\end{cases}$$

Uning grafigi 17-rasmda keltirilgan.

Endi ko‘rsatkichli taqsimotning matematik kutilmasi va dispersiyasini hisoblaymiz:

$$MX=\int_0^{+\infty} x\cdot\lambda e^{-\lambda x}dx=\lim_{b\to\infty}\int_0^b x\cdot\lambda e^{-\lambda x}dx=\lim_{b\to\infty}\left(-\int_0^b xde^{-\lambda x}\right)=$$

$$=\lim_{b\to\infty}\left(-x\cdot e^{-\lambda x}\Big|_0^b+\int_0^b e^{-\lambda x}dx\right)=\lim_{b\to\infty}\left(-\frac{1}{\lambda}e^{-\lambda x}\Big|_0^b\right)=\frac{1}{\lambda},$$

$$DX=\int_{-\infty}^{+\infty} x^2 f(x)dx-(MX)^2=\lambda\int_0^{+\infty} x^2\cdot e^{-\lambda x}dx-\frac{1}{\lambda^2}=$$

$$=[\text{bo'laklab integrallash formulasini ikki marta qo'llaymiz}]=$$

$$=\lambda\left(\lim_{b\to\infty}\left(-\frac{x^2}{\lambda}e^{-\lambda x}+\frac{2}{\lambda}\left(-\frac{x}{\lambda}e^{-\lambda x}-\frac{1}{\lambda^2}e^{-\lambda x}\right)\right)\Bigg|_0^b\right)-\frac{1}{\lambda^2}=\frac{2}{\lambda^2}-\frac{1}{\lambda^2}=\frac{1}{\lambda^2}.$$

Demak, agar $X\sim E(\lambda)$ bo‘lsa, u holda $MX=\frac{1}{\lambda}$ va $DX=\frac{1}{\lambda^2}$.

Normal taqsimot

Normal taqsimot ehtimollar nazariyasida o‘ziga xos o‘rin tutadi. Normal taqsimotning xususiyati shundan iboratki, u limit taqsimot hisoblanadi. Ya’ni boshqa taqsimotlar ma’lum shartlar ostida bu taqsimotga intiladi. Normal taqsimot amaliyotda eng ko‘p qo‘llaniladigan taqsimotdir.

✓ X uzluksiz t.m. *normal qonun* bo‘yicha taqsimlangan deyiladi, agar uning zichlik funksiyasi quyidagicha ko‘rinishga ega bo‘lsa

$$f(x)=\frac{1}{\sigma\cdot\sqrt{2\pi}}e^{-\frac{(x-a)^2}{2\sigma^2}},\ x\in R \qquad (2.6.6)$$

a va $\sigma>0$ parametrlar bo‘yicha normal taqsimot $N(a,\sigma)$ orqali belgilanadi. $X\sim N(a,\sigma)$ normal t.m.ning taqsimot funksiyasi

$$F(x)=\frac{1}{\sigma\cdot\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{(t-a)^2}{2\sigma^2}}dt. \qquad (2.6.7)$$

Agar normal taqsimot parametrlari a=0 va $\sigma=1$ bo‘lsa, u standart normal taqsimot deyiladi. Standart normal taqsimotning zichlik funksiyasi quyidagicha ko‘rinishga ega:

$$\varphi(x)=\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{x^2}{2}}.$$

Bu funksiya bilan 1.14 paragrafda tanishgan edik(uning grafigi 9-rasmda keltirilgan). Taqsimot funksiyasi

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-t^2/2}dt$$

ko‘rinishga ega va u Laplas funksiyasi deyiladi(uning grafigi 10-rasmda keltirilgan).

a va σ parametrlarni ma’nosini aniqlaymiz. Buning uchun $X \sim N(a,\sigma)$ t.m.ning matematik kutilmasi va dispersiyasini hisoblaymiz:

$$MX = \int_{-\infty}^{+\infty} x \cdot f(x)dx = \frac{1}{\sigma \cdot \sqrt{2\pi}} \int_{-\infty}^{+\infty} x \cdot e^{-\frac{(x-a)^2}{2\sigma^2}} dx = \left[\frac{x-a}{\sqrt{2}\sigma} = t \text{ almashtirish bajaramiz} \right] =$$

$$= \frac{1}{\sigma \cdot \sqrt{2\pi}} \int_{-\infty}^{+\infty} (\sqrt{2}\sigma t + a)e^{-t^2} \sqrt{2}\sigma dt = \frac{\sigma \cdot \sqrt{2}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} te^{-t^2} dt + \frac{a}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-t^2} dt = 0 + \frac{a}{\sqrt{\pi}} \cdot \sqrt{\pi} = a$$

Birinchi integral nolga teng, chunki integral ostidagi funksiya toq, integrallash chegarasi esa nolga nisbatan simmetrikdir. Ikkinchi integral esa Puasson integrali deyiladi,

$$\int_{-\infty}^{+\infty} e^{-t^2} dt = \sqrt{\pi}.$$

Shunday qilib, a parametr matematik kutilmani bildirar ekan. Dispersiya hisoblashda $\frac{x-a}{\sqrt{2}\sigma}$ =t almashtirish va bo‘laklab integrallashdan foydalanamiz:

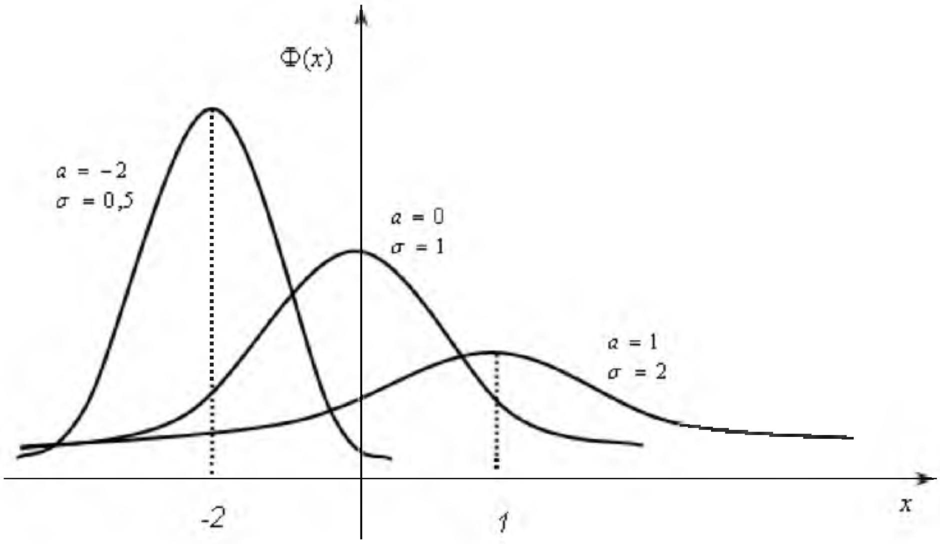
$$DX = \int_{-\infty}^{+\infty} (x-a)^2 \cdot f(x)dx = \frac{1}{\sigma \cdot \sqrt{2\pi}} \int_{-\infty}^{+\infty} (x-a)^2 \cdot e^{-\frac{(x-a)^2}{2\sigma^2}} dx =$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} 2\sigma^2 t^2 e^{-t^2} \sigma\sqrt{2}dt = \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} t^2 e^{-t^2} dt = \frac{2\sigma^2}{\sqrt{\pi}} \left(-\frac{1}{2} te^{-t^2} \Big|_{-\infty}^{+\infty} + \frac{1}{2} \int_{-\infty}^{+\infty} e^{-t^2} dt \right) =$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{1}{2} \sqrt{\pi} = \sigma^2.$$

Demak, $DX = \sigma^2$ va σ o‘rtacha kvadratik tarqoqlikni bildirar ekan.

18-rasmda a va σ larning turli qiymatlarida normal taqsimot grafigining o‘zgarishi tasvirlangan:



18-rasm.

$X \sim N(a,\sigma)$ t.m.ning (α,β) intervalga tushishi ehtimolligini hisoblaymiz. Avvalgi mavzulardan ma'lumki,

$$P\{\alpha < X < \beta\} = \int_{\alpha}^{\beta} f(x)dx = \frac{1}{\sigma\cdot\sqrt{2\pi}}\int_{\alpha}^{\beta} e^{-\frac{(x-a)^2}{2\sigma^2}}dx = \left[\frac{x-a}{\sqrt{2}\sigma}=\text{t}\right] = \frac{1}{\sqrt{2\pi}}\int_{\frac{\alpha-a}{\sigma}}^{\frac{\beta-a}{\sigma}} e^{-\frac{t^2}{2}}dt =$$

$$= \frac{1}{\sqrt{2\pi}}\int_{0}^{\frac{\beta-a}{\sigma}} e^{-\frac{t^2}{2}}dt - \frac{1}{\sqrt{2\pi}}\int_{0}^{\frac{\alpha-a}{\sigma}} e^{-\frac{t^2}{2}}dt.$$

Laplas funksiyasidan foydalanib((1.14.6) formula), quyidagiga ega bo'lamiz:

$$P\{\alpha < X < \beta\} = \Phi_0\left(\frac{\beta-a}{\sigma}\right) - \Phi_0\left(\frac{\alpha-a}{\sigma}\right). \tag{2.6.8}$$

Normal taqsimot taqsimot funksiyasini Laplas funksiyasi orqali quyidagicha ifodalasa bo'ladi:

$$F(x)=\int\limits_{-\infty}^{x}\frac{1}{\sigma\cdot\sqrt{2\pi}}\cdot e^{-\frac{(t-a)^2}{2\sigma^2}}dt=P\{-\infty<X<x\}=\Phi_0\left(\frac{x-a}{\sigma}\right)-\Phi_0\left(\frac{-\infty-a}{\sigma}\right)=$$

$$\Phi_0\left(\frac{x-a}{\sigma}\right)+\Phi_0(+\infty)=\Phi_0\left(\frac{x-a}{\sigma}\right)+\frac{1}{2} \qquad (2.6.9)$$

Agar Laplas funksiyasi $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{x}e^{-t^2/2}dt$ bo‘lsa, u holda $F(x)=\Phi\left(\frac{x-a}{\sigma}\right)$ va (2.6.8) formulani quyidagicha yozsa bo‘ladi:

$$P\{\alpha<X<\beta\}=\Phi\left(\frac{\beta-a}{\sigma}\right)-\Phi\left(\frac{\alpha-a}{\sigma}\right). \qquad (2.6.10)$$

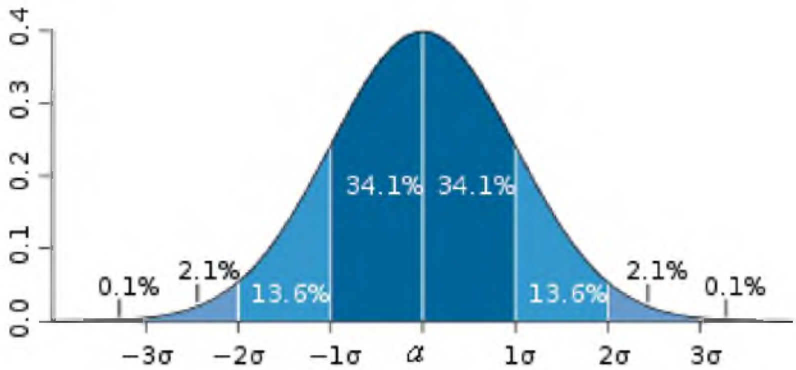
Amaliyotda ko‘p hollarda normal t.m.ning *a* ga nisbatan simmetrik bo‘lgan intervalga tushishi ehtimolligini hisoblashga to‘gri keladi. Uzunligi $2l$ bo‘lgan $(a\text{-}l,a+l)$ intervalni olaylik, u holda $P\{a-l\le X\le a+l\}=P\{|X-a|\le l\}=$

$$\Phi_0\left(\frac{a+l-a}{\sigma}\right)-\Phi_0\left(\frac{a-l-a}{\sigma}\right)=2\Phi_0\left(\frac{l}{\sigma}\right)=2\Phi\left(\frac{l}{\sigma}\right)-1.$$

Demak,

$$P\{|X-a|\le l\}=2\Phi_0\left(\frac{l}{\sigma}\right)=2\Phi\left(\frac{l}{\sigma}\right)-1. \qquad (2.6.11)$$

(2.6.11) da $l=3\sigma$ deb olsak, $P\{|X-a|\le 3\sigma\}=2\Phi_0(3)$ bo‘ladi. $\Phi_0(x)$ funksiyaning qiymatlari jadvalidan $\Phi_0(3)=0.49865$ ni topamiz. U holda $P\{|X-a|\le 3\sigma\}\approx 0.9973$ bo‘ladi. Bundan quyidagi muhim natijaga ega bo‘lamiz: Agar $X\sim N(a,\sigma)$ bo‘lsa, u holda uning matematik kutilishidan chetlashishining absolut qiymati o‘rtacha kvadratik tarqoqligining uchlanganidan katta bo‘lmaydi. Bu qoida “*uch sigma qoidasi*” deyiladi(19-rasm).



19-rasm.

2.7.-misol. Detallarni o'lchash jarayonida $\sigma = 10$ mm parametrli normal taqsimotga bo'ysuvuvchi tasodifiy xatoliklarga yo'l qo'yildi. Bog'liqsiz 3 marta detalni o'lchaganda hech bo'lmasa bitta o'lchash xatoligining absolut quymati 2 mm dan katta bo'lmasligi ehtimolligini baholang.

(2.6.11) formulaga ko'ra $P\{|X-a| \le 2\} = 2\Phi_0\left(\frac{2}{10}\right) \approx 2 \cdot 0.07926 = 0.15852.$ Bitta tajribada(o'lchashda) xatolikning 2 mm dan oshishi ehtimolligi $P\{|X-a| > 2\} = 1 - P\{|X-a| < 2\} \approx 0.84148$. Tajribalarimiz bog'liqsiz bo'lganligi uchun uchchala tajribada xatolikning 2 mm dan oshishi ehtimolligi $0.84148^3 \approx 0.5958$ bo'ladi. Qidirilayotgan ehtimollik 1-0.5958=0.4042.

II bobga doir misollar

1. Birinchi talabaning imtihonni topshira olishi ehtimolligi 0.6, ikkinchisiniki esa 0.9. Quyidagi hollar uchun imtihonni topshira olgan talabalar soni X t.m.ning taqsimot qonunini toping: a) Imtihonni qayta topshirish mumkin emas; b) imtihonni bir marta qayta topshirish mumkin.

2. A hodisaning ro'y berishi ehtimolligi 0.7 ga teng. Bog'liqsiz uchta tajribada A hodisaning ro'y berishlari soni X t.m.ning taqsimot qonunini toping.

3. Agar

X	1	2	3	4
P	0.3	0.2	0.4	0.1

bo‘lsa, X t.m.ning taqsimot funksiyasini toping.

4. Ikki ovchi bir nishonga qarata o‘q uzishmoqda. Birinchi ovchining nishonga tekkazishi ehtimolligi 0.6, ikkinchisiniki esa 0.8 bo‘lsa, nishonga tekkan o‘qlar soni X t.m.ning taqsimot qununini toping va taqsimot funksiyasini tuzing.

5. Taqsimot funksiyasi

$$F(x)=\begin{cases}0, \text{ agar } x<0,\\ 0.2, \text{ agar } 0\le x<1,\\ 0.6, \text{ agar } 1\le x<2,\\ 1, \text{ agar } x\ge 2\end{cases}$$

bo‘lgan X t.m.ning qabul qilishi mumkin bo‘lgan qiymatlari va ularga mos ehtimolliklarini toping.

6. Agar $P\{X>3\}=\dfrac{1}{3}$ bo‘lsa, $F_X(3)$ ni hisoblang.

7. Quyidagi funksiyalardan qaysilari zichlik funksiya bo‘ladi: $f_1(x)=-x^2$, $f_2(x)=\dfrac{1}{2}\sin x+\dfrac{1}{2}$, $f_3(x)=\dfrac{1}{\pi}\cdot\dfrac{1}{1+x^2}$.

8. Tasodifiy miqdorning zichlik funksiyasi

$$f(x)=\begin{cases}\dfrac{3}{2}x^2, & |x|\le h,\\ 0, & |x|>h,\end{cases}$$

bo‘lsa h ning qiymatini toping.

9. Taqsimot funksiyasi

$$F(x)=\begin{cases}0, \text{ agar } x\le 0,\\ \dfrac{x^2}{2}, \text{ agar, } 0<x\le\sqrt{2},\\ 1, \text{ agar } x>\sqrt{2},\end{cases}$$

bo‘lgan X t.m.ning zichlik funksiyasini toping va $P\{x<X<1\}$ ehtimollikni hisoblang.

10. Agar X t.m.ning taqsimot funksiyasi

$$F(x)=\begin{cases}0, \text{ agar } x\le -1,\\ a(x+1)^2, \text{ agar, -1} < x \le 2,\\ 1, \text{ agar } x>2\end{cases}$$

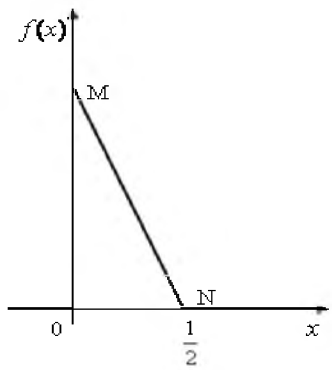
bo'lsa, o'zgarmas a ning qiymatini hisoblang.

11. Tasodifiy miqdorning zichlik funksiyasi

$$f(x)=\begin{cases}0, \text{ agar } x<-\dfrac{\pi}{2},\\ a\cos x, \text{ agar } -\dfrac{\pi}{2}\le x\le\dfrac{\pi}{2},\\ 0 \text{ agar } x>\dfrac{\pi}{2},\end{cases}$$

bo'lsa, o'zgarmas a ning qiymatini va t.m.ning taqsimot funksiyasini hisoblang.

12. Uzluksiz X t.m. zichlik funksiyasining grafigi berilgan:



20-rasm.

zichlik funksiya $f(x)$ ning ifodasini, $F(x)$ taqsimot funksiyasini va $\left\{1<X<\dfrac{1}{4}\right\}$ hodisaning ehtimolligini hisoblang.

13. $X \sim R(0,a)$ va $P\left\{X>\dfrac{1}{3}\right\}=\dfrac{1}{3}$ bo'lsa, a ning qiymatini toping.

14. Uzluksiz X t.m.ning zichlik funksiyasi:

$$f(x)=\begin{cases}0, & \text{agar } x \le 0 \text{ va } x > \pi,\\ \frac{1}{2}\sin x, & \text{agar } 0 < x < \pi,\end{cases}$$

bo'lsa, *MX* va *DX* ni hisoblang.

15. *X* va *Y* bog'liqsiz diskret t.m.lar bo'lib, *MX*=0, *MY*=-3, *DX*=2, *DY*=9 bo'lsa, *Z*=5*X*-3*Y*+2 t.m. uchun *MZ* va *DZ* ni hisoblang.

16. Uzluksiz *X* t.m.ning taqsimot qonuni:

$$F(x)=\begin{cases}0, & \text{agar } x \le A,\\ 0.25x^2, & \text{agar } A < x \le B,\\ 1, & \text{agar } x > B,\end{cases}$$

bo'lsa, *A* va *B* qiymatlarini toping, *MX* va σ_X ni hisoblang.

17. $X \sim N(3,2)$ bo'lsa, $P\{-3 < X < 5\}$, $P\{X \le 4\}$, $P\{|X-3| < 6\}$ ehtimolliklarni hisoblang.

18. Agar $X \sim Bi(1;0.5)$ bo'lsa, $(MX)^2$ va *DX* ni taqqoslang.

19. Quyida *X* t.m.ning taqsimot jadvali berilgan:

X	-0.5	0	0.5	1	1.5
P	0.1	0.4	0.1	0.3	0.1

a) $Y=10X-1$;

b) $Z=-X^2$

c) $V=2^X$ t.m.larning matematik kutilmasi va dispersiyaslarini hisoblang.

20. Agar $X \sim \Pi(23)$ va *Y*=1-*X* bo'lsa $F_Y(2)$ ni hisoblang.

21. Uzluksiz X t.m.ning zichlik funksiyasi quyidagicha ko'rinishga ega:

$$f(x)=\begin{cases}3h, & x \in [-1,0],\\ h, & x \in [0,2],\\ 0, & \text{aks holda.}\end{cases}$$

h ni, *X* t.m.ning taqsimot funksiyasi *F*(*x*) ni, *M*[(2-*X*)(*X*-3)] va *D*[2-3*X*] ni hisoblang.

22. *X* t.m.ning taqsimot funksiyasi

$$F(x)=\begin{cases}1-\frac{1}{x}, & \text{agar } x \ge 1,\\ 0, & \text{agar } x > 1,\end{cases}$$

bo‘lsa, $P\{X > a\} = \frac{1}{3}$ tenglik o‘rinli bo‘ladigan a ning qiymatini toping.

23. X uzluksiz t.m.ning taqsimot funksiyasi $F(x) = c + b\,\text{arctg}\frac{x}{a}$ formula orqali aniqlanadi. Quyidagilarni toping: a) o‘zgarmas a, b va c larning qiymatlari; b) X t.m.ning zichlik funksiyasi.

24. X t.m.ning taqsimot funksiyasi

$$F(x) = \begin{cases} 1 - e^{-2x}, & x \geq 0, \\ 0, & x < 0 \end{cases}$$

ko‘rinishga ega bo‘lsa, $M[(X-4)(5-X)], P\{X \leq MX\}$ va $D(3-2X)$ larni hisoblang.

25. Agar $f(x)$ zichlik funksiyasi bo‘lsa, u holda $f(-x)$ funksiya zichlik funksiya bo‘ladimi?

III bob. Ko‘p o‘lchovli tasodifiy miqdorlar

3.1 Ko‘p o‘lchovli tasodifiy miqdorlar va ularning birgalikdagi taqsimot funksiyasi

Bir o‘lchovli t.m.lardan tashqari, mumkin bo‘lgan qiymarlari 2 ta, 3 ta, ..., n ta son bilan aniqlanadigan miqdorlarni ham o‘rganish zarurati tug‘iladi. Bunday miqdorlar mos ravishda ikki o‘lchovli, uch o‘lchovli, … , *n* o‘lchovli deb ataladi.

Faraz qilaylik, $(\Omega, \mathcal{A}, P)$ ehtimollik fazosida aniqlangan $X_1, X_2, ..., X_n$ t.m.lar berilgan bo‘lsin.

✓ $X = (X_1, X_2, ..., X_n)$ vektorga tasodifiy vektor yoki *n*-o‘lchovli t.m. deyiladi.

Ko‘p o‘lchovli t.m. har bir elementar hodisa ω ga n ta $X_1, X_2, ..., X_n$ t.m.larning qabul qiladigan qiymatlarini mos qo‘yadi.

✓ $F_{X_1, X_2, ..., X_n}(x_1, x_2, ..., x_n) = P\{X_1 < x_1, X_2 < x_2, ..., X_n < x_n\}$ *n* o‘lchovli funksiya $X = (X_1, X_2, ..., X_n)$ tasodifiy vektorning taqsimot funksiyasi yoki $X_1, X_2, ..., X_n$ t.m.larning birgalikdagi taqsimot funksiyasi deyiladi.

Qulaylik uchun $F_{X_1, X_2, ..., X_n}(x_1, x_2, ..., x_n)$ taqsimot funksiyani $X_1, X_2, ..., X_n$ indekslarini tushirib qoldirib, $F(x_1, x_2, ..., x_n)$ ko‘rinishida yozamiz.

$F(x_1, x_2, ..., x_n)$ funksiya $X = (X_1, X_2, ..., X_n)$ tasodifiy vektorning taqsimot funksiyasi bo‘lsin. Ko‘p o‘lchovli $F(x_1, x_2, ..., x_n)$ taqsimot funksiyaning asosiy xossalarini keltiramiz:

1. $\forall x_i: \; 0 \le F(x_1, x_2, ..., x_n) \le 1$, ya’ni taqsimot funksiya chegaralangan.

2. $F(x_1, x_2, ..., x_n)$ funksiya har qaysi argumenti bo‘yicha kamayuvchi emas va chapdan uzluksiz.

3. Agar biror $x_i \to +\infty$ bo‘lsa, u holda

$$\lim_{x_i \to +\infty} F(x_1, x_2, ..., x_n) = F(x_1, ..., x_{i-1}, \infty, x_{i+1}, ..., x_n) = \\ = F_{X_1, ..., X_{i-1}, X_{i+1}, ..., X_n}(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) \quad (3.1.1)$$

4. Agar biror $x_i \to -\infty$ bo‘lsa, u holda $\lim_{x_i \to -\infty} F(x_1, x_2, ..., x_n) = 0$.

3-xossa yordamida keltirib chiqarilgan (3.1.1) taqsimot funksiyaga marginal(xususiy) taqsimot funksiya deyiladi. $X=(X_1,X_2,...,X_n)$ tasodifiy vektorning barcha marginal taqsimot funksiyalari soni $k=C_n^1+C_n^2+...+C_n^{n-1}=\sum_{n=0}^{n}C_n^m-C_n^0-C_n^n=2^n-2$ ga tengdir.

Masalan, $X=(X_1,X_2)$ (n=2) ikki o‘lchovlik tasodifiy vektorning marginal taqsimot funksiyalari soni $k=2^2-2=2$ ta bo‘lib, ular quyidagilardir: $F(x_1,+\infty)=F_1(x_1)=P(X_1<x_1)$; $F(+\infty,x_2)=F_2(x_2)=P(X_2<x_2)$.

Soddalik uchun n=2 bo‘lgan holda, ya’ni (X,Y) ikki o‘lchovlik tasodifiy vector bo‘lgan holni ko‘rish bilan cheklanamiz.

3.2 Ikki o‘lchovli diskret tasodifiy miqdor va uning taqsimot qonuni

(X,Y) ikki o‘lchovli t.m. taqsimot qonunini

$$p_{ij}=P\{X=x_i,Y=y_j\};\ i=\overline{1,n},\ j=\overline{1,m} \qquad (3.2.1)$$

formula yordamida yoki quyidagi jadval ko‘rinishida berish mumkin:

X \ Y	y_1	y_2	…	y_m
x_1	p_{11}	p_{12}	…	p_{1m}
x_2	p_{21}	p_{22}	…	p_{2m}
…	…	…	…	…
x_n	p_{n1}	p_{21}	…	p_{nm}

(3.2.2)

bu yerda barcha p_{ij} ehtimolliklar yig‘indisi birga teng, chunki $\{X=x_i,Y=y_j\}$ $i=\overline{1,n},\ j=\overline{1,m}$ birgalikda bo‘lmagan hodisalar to‘la gruppani tashkil etadi $\sum_{i=1}^{n}\sum_{j=1}^{m}p_{ij}=1$. (3.2.1) formula ikki o‘lchovli diskret t.m.ning taqsimot qonuni, (3.2.2) jadval esa birgalikdagi taqsimot jadvali deyiladi.

(X,Y) ikki o‘lchovli diskret t.m.ning birgalikdagi taqsimot qonuni berilgan bo‘lsa, har bir komponentaning alohida (marginal) taqsimot qonunlarini topish mumkin. Har bir $i=\overline{1,n}$ uchun

$\{X=x_i, Y=y_1\}, \{X=x_i, Y=y_2\}, ..., \{X=x_i, Y=y_m\}$ hodisalar birgalikda bo‘lmagani sababli: $p_{x_i}=P\{X=x_i\}=p_{i1}+p_{i2}+...+p_{im}$. Demak, $p_{x_i}=P\{X=x_i\}=\sum_{j=1}^{m}p_{ij}$, $i=\overline{1,n}$, $p_{y_j}=P\{Y=y_j\}=\sum_{i=1}^{n}p_{ij}$ $j=\overline{1,m}$.

3.1-misol. Ichida 2 ta oq, 1 ta qora, 1 ta ko‘k shar bo‘lgan idishdan tavakkaliga ikkita shar olinadi. Olingan sharlar ichida qora sharlar soni X t.m. va ko‘k rangdagi sharlar soni Y t.m. bo‘lsin. (X,Y) ikki o‘lchovli t.m.ning birgalikdagi taqsimot qonunini tuzing. X va Y t.m.larning alohida taqsimot qonunlarini toping.

X t.m. qabul qilishi mumkin qiymatlari: 0 va 1: Y t.m.ning qiymatlari ham 0 va 1. Mos ehtimolliklarni hisoblaymiz:

$p_{11}=P\{X=0,Y=0\}=\frac{C_2^2}{C_4^2}=\frac{1}{6}$ (yoki $\frac{2}{4}\cdot\frac{1}{3}=\frac{1}{6}$);

$p_{12}=P\{X=0,Y=1\}=\frac{C_2^1}{C_4^2}=\frac{2}{6}$; $p_{21}=P\{X=1,Y=0\}=\frac{2}{6}$;

$p_{22}=P\{X=1,Y=1\}=\frac{1}{6}$.

(X,Y) vaktorning taqsimot jadvali quyidagicha ko‘rinishga ega:

X \ Y	0	1
0	$\frac{1}{6}$	$\frac{2}{6}$
1	$\frac{2}{6}$	$\frac{1}{6}$

Bu yerdan $P\{X=0\}=\frac{1}{6}+\frac{2}{6}=\frac{1}{2}$, $P\{X=1\}=\frac{2}{6}+\frac{1}{6}=\frac{1}{2}$; $P\{Y=0\}=\frac{1}{6}+\frac{2}{6}=\frac{1}{2}$, $P\{Y=1\}=\frac{2}{6}+\frac{1}{6}=\frac{1}{2}$ kelib chiqadi. X va Y t.m.larning alohida taqsimot qonunlari quyidagi ko‘rinishga ega bo‘ladi:

$$\begin{cases} X: & 0, & 1 \\ p: & \frac{1}{2}, & \frac{1}{2} \end{cases} \text{ va } \begin{cases} Y: & 0, & 1 \\ p: & \frac{1}{2}, & \frac{1}{2} \end{cases}.$$

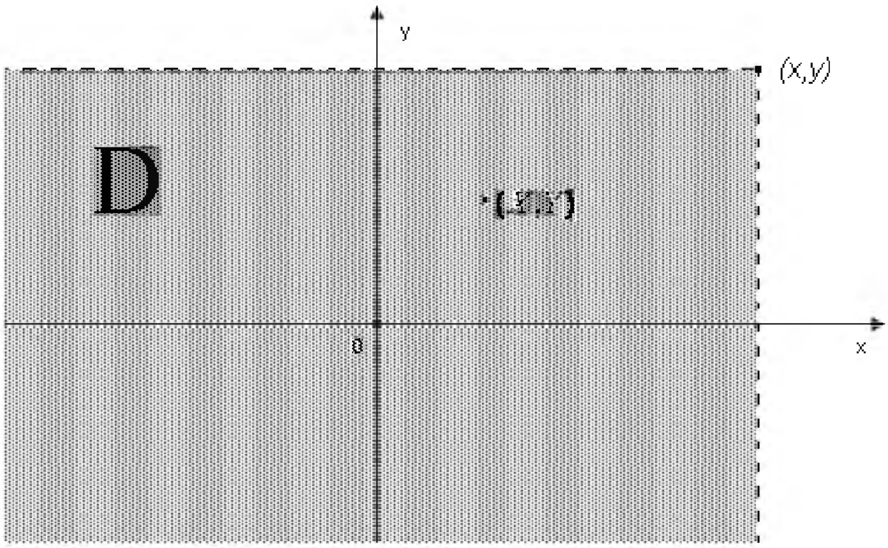
3.3 Ikki o‘lchovli tasodifiy miqdorning taqsimot funksiyasi va uning xossalari

Ikki o‘lchovli t.m. taqsimot funksiyasini $F(x,y)$ orqali belgilaymiz.

✓ *Ikki o‘lcholi (X,Y) t.m.ning taqsimot funksiyasi*, x va y sonlarning har bir jufti uchun $\{X\le x\}$ va $\{Y\le y\}$ hodisalarning birgalikdagi ehtimolligini aniqlaydigan $F(x,y)$ funksiyasidir: ya’ni

$$F(x,y)=P\{X\le x,Y\le y\}=P\big((X.Y)\in(-\infty,x)\times(-\infty,y)=D\big). \qquad (3.3.1)$$

(3.3.1.) tenglikning geometrik tasviri 21-rasmda keltirilgan.



21-rasm.

(X,Y) ikki o‘lchovlik diskret t.m. taqsimot funksiyasi quyidagi yig‘indi orqali aniqlanadi:

$$F(x,y)=\sum_{x_i<x}\sum_{y_j<y}p_{ij}. \qquad (3.3.2)$$

Ikki o‘lchovlik t.m. taqsimot funksiyasining xossalari:

1. $F(x,y)$ taqsimot funksiya chegaralangan: $0\le F(x,y)\le 1$.

2. $F(x,y)$ funksiya har qaysi argumenti bo‘yicha kamayuvchi emas:

agar $x_2>x_1$ bo‘lsa, $F(x_2,y)\ge F(x_1,y)$,

agar $y_2>y_1$ bo‘lsa, $F(x,y_2)\ge F(x,y_1)$.

3. $F(x,y)$ funksiyaning biror argumenti $-\infty$ bo‘lsa(limit ma’nosida), u holda $F(x,y)$ funksiya nolga teng, $F(x,-\infty)=F(-\infty,y)=F(-\infty,-\infty)=0$.

4. Agar $F(x,y)$ funksiyaning bitta argumenti $+\infty$ bo‘lsa(limit ma’nosida), u holda

$$F(x,+\infty)=F_1(x)=F_X(x);\ F(+\infty,y)=F_2(y)=F_Y(y). \qquad (3.3.3)$$

$4'$. Agar ikkala argumenti $+\infty$ bo'lsa(limit ma'nosida), u holda $F(+\infty,+\infty)=1$.

5. $F(x,y)$ funksiya har qaysi argumenti bo'yicha chapdan uzluksiz, ya'ni $\lim\limits_{x\to x_0-0} F(x,y)=F(x_0,y)$, $\lim\limits_{y\to y_0-0} F(x,y)=F(x,y_0)$.

Isboti. 1. $F(x,y)=P\{X\le x,Y\le y\}$ ehtimollik bo'lgaligi uchun $0\le F(x,y)\le 1$.

2. (x,y) argumentlarning birortasini kattalashtirsak, 21-rasmda bo'yalgan D soha kattalashadi, demak bu sohaga (X,Y) tasodifiy nuqtaning tushishi ehtimolligi kamaymaydi.

3. $\{X<-\infty\},\{Y<-\infty\}$ hodisalar va ularning ko'paytmasi mumkin bo'lmagan hodisalardir. Demak, bu hodisalarning ehtimolligi nolga teng.

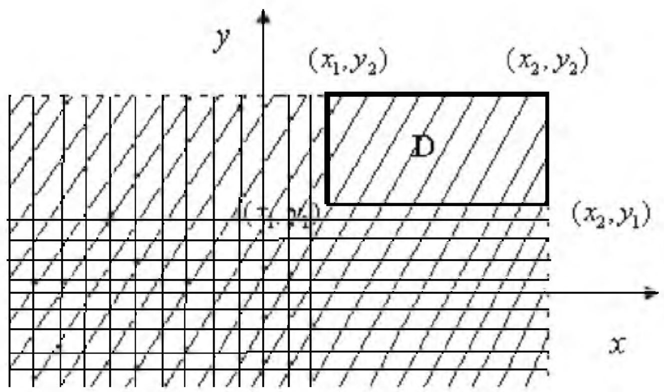
4. $\{X<+\infty\}$ muqarrar hodisa bo'lgani uchun $\{X<+\infty\}\cdot\{Y\le y\}=\{Y\le y\}$ bo'ladi. Demak, $F(+\infty,y)=P\{X<+\infty;Y\le y\}=P\{Y\le y\}=F_Y(y)$. Xuddi shunday $F(x,+\infty)=P\{X\le x;Y<+\infty\}=P\{X\le x\}=F_X(x)$.

$4'$. $\{X<+\infty\}$ va $\{Y<+\infty\}$ hodisalar muqarrar hodisalar bo'lganligi uchun $\{X<+\infty\}\cdot\{Y<+\infty\}$ ham muqarrar hodisa bo'ladi va bu hodisaning ehtimolligi 1 ga teng. ■

$F(x,y)$ taqsimot funksiya yordamida (X,Y) t.m. biror $D=\{(x,y):x_1\le x\le x_2,y_1\le y\le y_2\}$ sohaga tushishi ehtimolligini topish mumkin:

$$P\{(X,Y)\in D\}=P\{x_1\le X\le x_2,y_1\le Y\le y_2\}= \\ =F(x_2,y_2)-F(x_1,y_2)-F(x_2,y_1)+F(x_1,y_1). \quad (3.3.4)$$

22-rasmda (3.3.4) tenglikning geometrik isboti keltirilgan.



22-rasm.

3.2-misol. 3.1-misoldagi (X,Y) ikki o'lchovlik t.m.ning hamda X va Y t.m.larning taqsimot funksiyalarini toping.
Avvalgi bobdagi (2.3.2) formuladan:

$$F_1(x)=\begin{cases}0, \text{ agar } x\le 0,\\ 0.5, \text{ agar } 0<x\le 1,\\ 1, \text{ agar } x>1,\end{cases} \qquad F_2(y)=\begin{cases}0, \text{ agar y} \le 0,\\ 0.5, \text{ agar } 0<y\le 1,\\ 1, \text{ agar y} >1.\end{cases}$$

(X,Y) ikki o'lchovlik t.m.ning $F(x,y)$ taqsimot funksiyasini (3.3.2) formulaga ko'ra topamiz:

X \ Y	$y\le 0$	$0<y\le 1$	$y>1$
$x\le 0$	0	0	0
$0<x\le 1$	0	$\frac{1}{6}$	$\frac{1}{2}\left(=\frac{1}{6}+\frac{2}{6}\right)$
$x>1$	0	$\frac{1}{2}\left(=\frac{1}{6}+\frac{2}{6}\right)$	$1\left(=\frac{1}{6}+\frac{2}{6}+\frac{2}{6}+\frac{1}{6}\right)$

3.4 Ikki o'lchovlik uzluksiz tasodifiy miqdor zichlik funksiyasi va uning xossalari

✓ Ikki o'lchovlik t.m. uzluksiz deyiladi, agar uning taqsimot funksiyasi $F(x,y)$: 1. uzluksiz bo'lsa;

2. har bir argumenti bo'yicha differensiyallanuvchi;
3. $F''_{xy}(x,y)$ ikkinchi tartibli aralash hosila mavjud bo'lsa.

✓ *Ikki o'lchovlik (X,Y) t.m.ning zichlik funksiyasi*

$$f(x,y)=\frac{\partial^2 F(x,y)}{\partial x\partial y}=F''_{xy}(x,y) \qquad (3.4.1)$$

Tenglik orqali aniqlanadi.

(X,Y) t.m.ning G sohaga(23-rasm) tushishi ehtimolligi (3.3.4) formulaga ko'ra: $P\{x\le X<x+\vartriangle x, y\le Y<y+\vartriangle y\}=$

$$= F(x+\Delta x, y+\Delta y) - F(x, y+\Delta y) - F(x+\Delta x, y) + F(x, y),$$

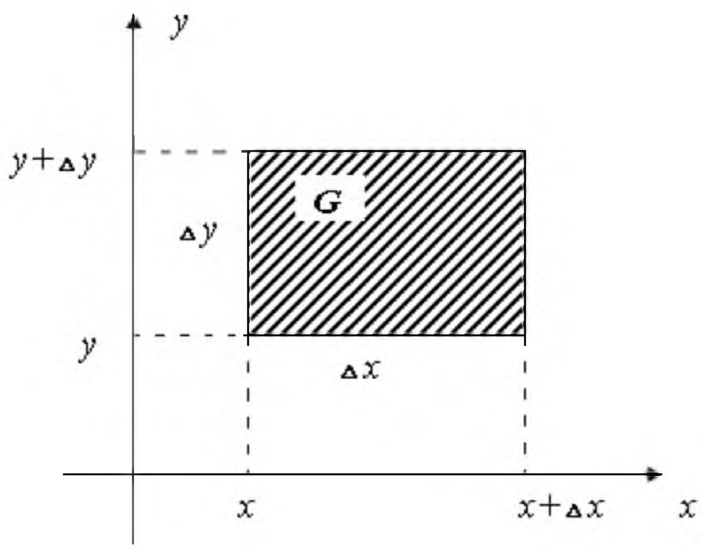
$$f_{\text{o'rtacha}} = \frac{P\{x \le X < x+\Delta x, y \le Y < y+\Delta y\}}{\Delta x \cdot \Delta y} =$$

$$= \frac{1}{\Delta y}\left(\frac{F(x+\Delta x, y+\Delta y) - F(x, y+\Delta y)}{\Delta x} - \frac{F(x+\Delta x, y) - F(x, y)}{\Delta x}\right).$$

$\Delta x \to 0, \Delta y \to 0$ da limitga o‘tamiz,

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} f_{\text{o'rtacha}} = \lim_{\Delta y \to 0} \frac{F'_x(x, y+\Delta y) - F'_x(x, y)}{\Delta y},$$

ya’ni $f(x, y) = \left(F'_x(x, y)\right)'_y = F''_{xy}(x, y)$.



23-rasm.

Demak, (X,Y) ikki o‘lchovli tasodifiy vektorning zichlik funksiyasi deb,

$$P\{x \le X < x+dx, y \le Y < y+dy\} \approx f(x, y)dxdy \qquad (3.4.2)$$

tenglikni qanoatlantiruvchi funksiya ekan.

$f(x,y)$ zichlik funkiyasi quyidagi xossalarga ega:

1. $f(x,y) \geq 0$.

2. $P\{(X,Y) \in D\} = \iint_D f(x,y)dxdy$. (3.4.3)

3. $F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)dudv$. (3.4.4)

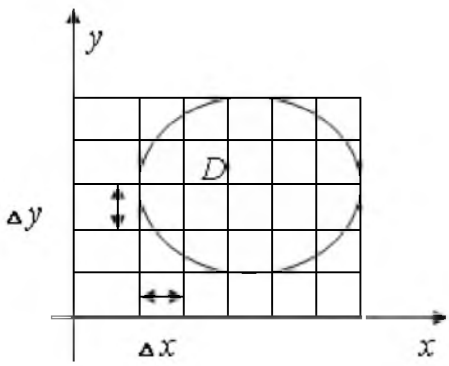
4. $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)dxdy = 1$.

5. X va Y t.m.larning bir o‘lchovlik zichlik funksiyalarini quyidagi tengliklar yordamida topish mumkin:

$$\int_{-\infty}^{+\infty} f(x,y)dy = f_X(x); \quad \int_{-\infty}^{+\infty} f(x,y)dx = f_Y(y). \qquad (3.4.5)$$

Isboti. 1. Bu xossa $F(x,y)$ funksiyaning har qaysi argumenti bo‘yicha kamaymaydigan funksiya ekanligidan kelib chiqadi.

2. $f(x,y)dxdy$ ifoda (X,Y) tasodifiy nuqtaning tomonlari dx va dy bo‘lgan to‘g‘ri to‘rtburchakka tushish ehtimolligini bildiragi. D sohani to‘g‘ri to‘rtburchaklarga ajratamiz(24-rasm) va har biri uchun (3.4.2) formulani qo‘llaymiz:



24-rasm.

$P\{(X,Y) \in D\} \approx \sum_{i=1}^{n} f(x_i,y_i)\Delta x\Delta y$ bo‘ladi. Endi $\Delta x \to 0, \Delta y \to 0$ da limitga o‘tib, $P\{(X,Y) \in D\} = \iint_D f(x,y)dxdy$ ni hosil qilamiz.

3. (3.4.3) formuladan:

$$F(x,y) = P\{X < x, Y < y\} = P\{-\infty < X < x, -\infty < Y < y\} = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)dudv.$$

4. $F(+\infty,+\infty)=1$ va (3.4.4) formulada $x=y=+\infty$ deb olsak(limit ma'nosida),

$$F(+\infty,+\infty)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(x,y)dxdy=1.$$

5. Avval X va Y t.m.larning taqsimot funksiyalarini topamiz:

$$F_X(x)=F(x,+\infty)=\int_{-\infty}^{x}\int_{-\infty}^{+\infty}f(u,v)dudv=\int_{-\infty}^{x}\left(\int_{-\infty}^{+\infty}f(u,y)dy\right)du,$$

$$F_Y(y)=F(+\infty,y)=\int_{-\infty}^{+\infty}\int_{-\infty}^{y}f(u,v)dudv=\int_{-\infty}^{y}\left(\int_{-\infty}^{+\infty}f(x,v)dx\right)dv. \qquad (3.4.5)$$

Birinchi tenglikni x bo'yicha, ikkinchisini y bo'yicha differensiyallasak, X av Y t.m.larnin zichlik funksiyalarini hosil qilamiz:

$$f_X(x)=F_X^{'}(x)=\int_{-\infty}^{+\infty}f(x,y)dy$$

va

$$f_Y(y)=F_Y^{'}(y)=\int_{-\infty}^{+\infty}f(x,y)dx.$$

■

Izoh. Agar X va Y t.m.larning alohida zichlik funksiyalari berilgan bo'lsa, (umumiy holda) ularning birgalikdagi zichlik funksiyalarini topish mumkin emas.

3.3-misol. (X,Y) ikki o'lchovli t.m.ning birgalidagi zichlik funksiyasi berilgan

$$f(x,y)=\begin{cases}Ce^{-x-y}, & \text{agar } x\geq 0, y\geq 0\\ 0, & \text{aks holda.}\end{cases}$$

Quyidagilarni toping: 1) O'zgarmas son C; 2) $F(x,y)$; 3) $F_X(x)$ va $F_Y(y)$; 4) $f_X(x)$ va $f_Y(y)$; 5) $P\{X>0,Y<1\}$.

1) $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)dxdy = 1$ tenglikdan

$$C\int_{0}^{+\infty}\int_{0}^{+\infty} e^{-x-y}dxdy = C\int_{0}^{+\infty} e^{-x}dx \cdot \int_{0}^{+\infty} e^{-y}dy = C = 1.$$

2) $F(x,y) = \int_{0}^{x}\int_{0}^{y} e^{-u-v}dudv = \int_{0}^{x} e^{-u}du \cdot \int_{0}^{y} e^{-v}dv = (1-e^{-x})(1-e^{-y})$, $x \geq 0, y \geq 0$,

ya'ni

$$F(x,y) = \begin{cases} (1-e^{-x})(1-e^{-y}), & x \geq 0, y \geq 0, \\ 0, & \text{aks holda.} \end{cases}$$

3) $F_X(x) = F(x,+\infty) = \int_{0}^{x}\left(\int_{0}^{+\infty} e^{-u}e^{-v}dv\right)du = \int_{0}^{x} 1 \cdot e^{-u}du = 1 - e^{-x}$, $x \geq 0$, demak

$$F_X(x) = \begin{cases} (1-e^{-x}), & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Aynan shunday,

$$F_Y(x) = \begin{cases} (1-e^{-y}), & y \geq 0, \\ 0, & y < 0. \end{cases}$$

4) $f_X(x) = F_X^{'}(x) = \begin{cases} (1-e^{-x})_x^{'}, & x \geq 0, \\ 0, & x < 0, \end{cases} = \begin{cases} e^{-x}, & x \geq 0, \\ 0, & x < 0, \end{cases}$

va shu kabi

$$f_Y(y) = \begin{cases} e^{-y}, & y \geq 0, \\ 0, & y < 0. \end{cases}$$

5) $P\{X > 0, Y < 1\} = \int_{0}^{\infty} e^{-x}dx \int_{0}^{1} e^{-y}dy = -(e^{-1}-1)\int_{0}^{\infty} e^{-x}dx = 1 - \frac{1}{e} \approx 0.63.$

3.5 Tasodifiy miqdorlarning bog‘liqsizligi

✓ X va Y t.m.lar *bog‘liqsiz* deiladi, agar $\forall x, y \in R$ uchun $\{X < x\}$ va $\{Y < y\}$ hodisalar bog‘liqsiz bo‘lsa.

Endi t.m.lar bog‘liqsizligining zarur va yetarli shartini keltiramiz.

Teorema. X va Y t.m.lar bog‘liqsiz bo‘lishi uchun

$$F(x, y) = F_X(x) \cdot F_Y(y) \qquad (3.5.1)$$

tenglik bajarilishi zarur va yetarlidir.

Isboti. Zarurligi. Agar X va Y t.m.lar bog‘liqsiz bo‘lsa, $\{X < x\}$ va $\{Y < y\}$ hodisalar ham bog‘liqsiz bo‘ladi. U holda $P\{X < x, Y < y\} = P\{X < x\} \cdot P\{Y < y\}$, ya’ni $F(x, y) = F_X(x) \cdot F_Y(y)$.

Yetarliligi. (3.5.1) tenglik o‘rinli bo‘lsin, u holda $P\{X < x, Y < y\} = P\{X < x\} \cdot P\{Y < y\}$ bo‘ladi. Bu tenglikdan X va Y t.m.lar bog‘liqsizligi kelib chiqadi. ■

1-natija. X va Y uzluksiz t.m.lar bog‘liqsiz bo‘lishi uchun

$$f(x, y) = f_X(x) \cdot f_Y(y) \qquad (3.5.2)$$

tenglik bajarilishi zarur va yetarlidir.

Isboti. Zarurligi. Agar X va Y t.m.lar bog‘liqsiz bo‘lsa, u holda (3.5.1) tenglik o‘rinli bo‘ladi. Bu tenglikni x bo‘yicha, keyin esa y bo‘yicha differensiyallab, $f(x, y) = \frac{d}{dx} F_X(x) \cdot \frac{d}{dy} F_Y(y)$ tengliklarni, ya’ni $f(x, y) = f_X(x) \cdot f_Y(y)$ hosil qilamiz.

Yetarliligi. (3.5.2) tenglik o‘rinli bo‘lsin. Bu tenglikni x bo‘yicha va y bo‘yicha integrallaymiz:

$$\int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v)\,du\,dv = \int_{-\infty}^{x} f_X(u)\,du \cdot \int_{-\infty}^{y} f_Y(v)\,dv.$$

Bu esa $F(x, y) = F_X(x) \cdot F_Y(y)$ tenglikning o‘zidir. Teoremaga ko‘ra X va Y t.m.lar bog‘liqsizligi kelib chiqadi. ■

2-natija. X va Y diskret t.m.lar bog‘liqsiz bo‘lishi uchun ihtiyoriy $i=1,2,...n,\ j=1,2,...m$ larda

$$P\{X=x_i, Y=y_j\}=P\{X=x_i\}\cdot P\{Y=y_j\} \qquad (3.5.3)$$

tengliklarning bajarilishi zarur va yetarlidir.

3.4-misol. a) 3.1-misoldagi X va Y t.m.lar bog‘liqmi? b) 3.3-misoldagi X va Y t.m.lar bog‘liqsizmi?

a) $p_{11}=P\{X=0,Y=0\}=\frac{1}{6}$, $P\{X=0\}\cdot P\{Y=0\}=\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$, ya’ni $P\{X=0,Y=0\}\neq P\{X=0\}\cdot P\{Y=0\}$. Demak, X va Y t.m.lar bog‘liq.

b) $f(x,y)=\begin{cases} e^{-x-y}, & \text{agar } x\geq 0, y\geq 0 \\ 0, & \text{aks holda,}\end{cases}$ $\quad f_X(x)=\begin{cases} e^{-x}, & x\geq 0, \\ 0, & x<0,\end{cases}$

$f_Y(y)=\begin{cases} e^{-y}, & y\geq 0, \\ 0, & y<0.\end{cases}$ $f(x,y)=f_X(x)\cdot f_Y(y)$ tenglik o‘rinli, demak, X va Y t.m.lar bog‘liqsiz.

3.6 Shartli taqsimot qonunlari

(X,Y) ikki o‘lchovlik t.m.ni tashkil etuvchi X va Y t.m.lar bog‘liq bo‘lsa, ularning bog‘liqligini xarakterlovchi shartli taqsimot qonunlari tushunchalari keltiriladi.

✓ (X,Y) ikki o‘lchovli diskret t.m. birgalikdagi taqsimot qonuni $p_{ij}=P\{X=x_i, Y=y_j\}$, $i=\overline{1,n},\ j=\overline{1,m}$ bo‘lsin. U holda

$$P\{Y=y_j / X=x_i\}=\frac{P\{X=x_i, Y=y_j\}}{P\{X=x_i\}},\ i=1,2,...n,\ j=1,2,...m \qquad (3.6.1)$$

ehtimolliklar to‘plami, ya’ni $p(y_1/x_i), p(y_2/x_i),...p(y_m/x_i)$ lar Y t.m.ning $X=x_i$ dagi *shartli taqsimot qonuni* deyiladi. Bu yerda

$$\sum_{j=1}^{m} p(y_j/x_i)=\sum_{j=1}^{m}\frac{p_{ij}}{p_{x_i}}=\frac{1}{p_{x_i}}\sum_{j=1}^{m} p_{ij}=\frac{p_{x_i}}{p_{x_i}}=1.$$

Xuddi shunday,

$$P\{X=x_i/Y=y_j\}=\frac{P\{X=x_i,Y=y_j\}}{P\{Y=y_i\}},\ i=1,2,...n,\ j=1,2,...m \qquad (3.6.2)$$

ehtimolliklar to‘plami, ya’ni $p(x_1/y_j), p(x_2/y_j),...p(x_n/y_j)$ lar X t.m.ning $Y=y_j$ dagi shartli taqsimot qonuni deyiladi.

3.5-misol. (X,Y) ikki o‘lchovlik t.m.ni birgalikdagi taqsimot jadvali berilgan:

$X\setminus Y$	1	2	3
0.1	0.12	0.08	0.40
0.2	0.16	0.10	0.14

Quyidagilarni toping: a) X av Y t.m.larning alohida taqsimot qonunlari; b) X t.m.ning Y=2 dagi shartli taqsimot qonuni.

a) $p_{x_i}=\sum_{j=1}^{m}p_{ij}$ va $p_{y_j}=\sum_{i=1}^{n}p_{ij}$ tengliklardan:

X	0.1	0.2
P	0.60	0.40

,

Y	1	2	3
P	0.28	0.10	0.54

b) (3.6.2) formulaga asosan: $P\{X=0.1/Y=2\}=\frac{0.08}{0.18}=\frac{4}{9}$,

$P\{X=0.2/Y=2\}=\frac{0.10}{0.18}=\frac{5}{9}$. X t.m.ning Y=2 dagi shartli taqsimot qonuni quyidagiga teng:

X	0.1	0.2
$P_{Y=2}$	$\frac{4}{9}$	$\frac{5}{9}$

Endi (X,Y) ikki o‘lchovli t.m. uzluksiz bo‘lgan holni ko‘ramiz. $f(x,y)$ (X,Y) t.m.ning birgalikdagi zichlik funksiyasi, $f_X(x)$ va $f_Y(y)$ lar esa X va Y t.m.larning alohida zichlik funksiyalari bo‘lsin.

- ✓ Y t.m.ning X=x bo‘lgandagi *shartli zichlik funksiyasi*

$$f(y/x)=\frac{f(x,y)}{f_X(x)}=\frac{f(x,y)}{\int_{-\infty}^{+\infty}f(x,y)dy},\ f_X(x)\neq 0 \qquad (3.6.3)$$

ifodaga orqali aniqlanadi.

Shartli zichlik funksiyasi zichlik funksiyasining $f(y/x) \geq 0, \int_{-\infty}^{+\infty} f(y/x)dy = 1$ kabi xossalariga egadir.

✓ Xuddi shunday, X t.m.ning $Y=y$ bo'lgandagi shartli zichlik funksiyasi

$$f(x/y) = \frac{f(x,y)}{f_Y(y)} = \frac{f(x,y)}{\int_{-\infty}^{+\infty} f(x,y)dx}, \ f_y(y) \neq 0, \qquad (3.6.4)$$

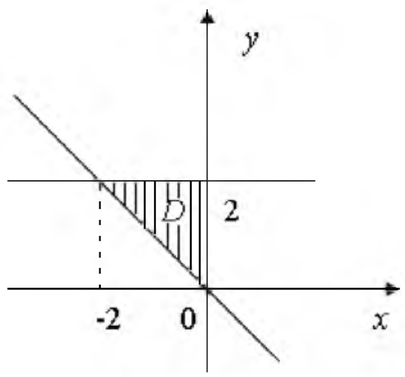
tenglik orqali aniqlanadi.

(3.6.3) va (3.6.4) tengliklarni hisobga olib, $f(x,y)$ zichlik funksiyani quyidagi ko'rinishda yozish mumkin:

$$f(x,y) = f_X(x) \cdot f(y/x) = f_Y(y) \cdot f(x/y). \qquad (3.6.5)$$

(3.6.5) tenglik zichlik funksiyalarning ko'paytirish qoidasi(teoremasi) deyiladi.

3.6-misol. (X,Y) ikki o'lchovli uzluksiz t.m.ning birgalikdagi zichlik funksiyasi berilgan: $f(x,y) = \begin{cases} Cxy, & \text{agar } (x,y) \in D, \\ 0, & \text{agar } (x,y) \notin D, \end{cases}$



25-rasm.

bu yerda $D = \{(x,y): y > -x, y < 2, x < 0\}$ (25-rasm). 1) $f_X(x)$ va $f(x/y)$ larni toping. 2) X va Y t.m.larning bog'liqligini ko'rsating.

1) Avval o'zgarmas son C ni topamiz:

$$1=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)dxdy=\int_{-2}^{0}dx\int_{-x}^{2}Cxydy=C\int_{-2}^{0}xdx\left(\frac{y^2}{2}\Big|_{-x}^{2}\right)=C\int_{-2}^{0}x\left(2-\frac{x^2}{2}\right)dx=-2C.$$

Bundan $C=\frac{1}{2}$. $f_X(x)$ ni topamiz:

$$f_X(x)=\int_{-\infty}^{+\infty} f(x,y)dy=\int_{-2}^{0}\left(-\frac{1}{2}xy\right)dy=-\frac{1}{4}x(4-x^2),\ x\in(-2,0).$$

$f(x/y)$ ni (3.6.4) formulasidan foydalanamiz, buning uchun dastlab $f_Y(y)$ ni hisoblash kerak:

$$f_Y(y)=\int_{-\infty}^{+\infty} f(x,y)dx=\int_{-y}^{0}\left(-\frac{1}{2}xy\right)dx=\frac{y^3}{4},\ y\in(0,2),$$

$$f(x/y)=\frac{f(x,y)}{f_Y(y)}=\frac{-\frac{1}{2}xy}{\frac{y^3}{4}}=-\frac{2x}{y^2},\ (x,y)\in D.$$

2) X va Y t.m.lar bog‘liqsiz bo‘lsa, $f(x/y)=\frac{f(x,y)}{f_Y(y)}=\frac{f_X(x)\cdot f_Y(y)}{f_Y(y)}=f_X(x)$ tenglik o‘rinli. $f_X(x)=-\frac{1}{4}x(4-x^2)$, $x\in(-2,0)$ va $f(x/y)=-\frac{2x}{y^2}$, $(x,y)\in D$ funksiyalarlar bir-biridan farqli bo‘lganligi uchun X va Y t.m.lar bog‘liq.

3.7 Ikki o‘lchovli tasodifiy miqdorlarning sonli xarakteristikalari

(X,Y) tasodifiy vektorning sonli xarakteristikalari sifatida turli tartibdagi momentlar ko‘riladi. Amaliyotda eng ko‘p I va II – tartibli momentlar bilan ifodalanuvchi matematik kutilma, dispersiya va korrelatsion momentlardan foydalaniladi.

✓ Ikki o‘lchovli diskret (X,Y) t.m.ning *matematik kutilmasi* (MX,MY) bo‘lib, bu yerda

$$MX = m_x = \sum_{i=1}^{n}\sum_{j=1}^{m} x_i p_{ij}, \quad MY = m_y = \sum_{i=1}^{n}\sum_{j=1}^{m} y_i p_{ij} \tag{3.7.1}$$

va $p_{ij} = P\{X = x_i, Y = y_j\}$.

Agar (X,Y) t.m. uzluksiz bo'lsa, u holda

$$MX = m_x = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x \cdot f(x,y)dxdy, \quad MY = m_y = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y \cdot f(x,y)dxdy. \tag{3.7.2}$$

✓ X va Y t.m.larning *kovariatsiyasi*

$$K_{XY} = \mathrm{cov}(X,Y) = M\big((X - m_x)(Y - m_y)\big) = \mu_{1,1} \tag{3.7.3}$$

tenglik bilan aniqlanadi. Agar (X,Y) t.m. diskret bo'lsa, uning kovariatsiyasi

$$K_{XY} = \sum_{i=1}^{n}\sum_{j=1}^{m} (x_i - m_x)(y_j - m_y)p_{ij}, \tag{3.7.4}$$

agar uzluksiz bo'lsa,

$$K_{XY} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x - m_x)(y - m_y)f(x,y)dxdy \tag{3.7.5}$$

formulalar orqali hisoblanadi.

Kovariatsiyani quyidagicha hisoblash ham mumkin:

$$K_{XY} = \mathrm{cov}(X,Y) = MXY - MX \cdot MY. \tag{3.7.6}$$

Bu tenglik (3.7.3) formula va matematik kutilmaning xossalaridan kelib chiqadi:

$$K_{XY} = M\big((X - m_x)(Y - m_y)\big) = M(XY - Xm_y - Ym_x + m_x m_y) =$$

$= MXY - m_y MX - m_x MY + m_x m_y = MXY - m_y m_x - m_x m_y + m_x m_y = MXY - MXMY.$

Kovariatsiya orqali X va Y t.m.larning dispersiyalarini aniqlash mumkin:

$DX = \mathrm{cov}(X,X) = M(X - MX)^2 = MX^2 - (MX)^2,$

$DY = \mathrm{cov}(Y,Y) = M(Y - MY)^2 = MY^2 - (MY)^2.$

(X,Y) vektorning kovariatsiya matritsasi

$C = M\left\{(X,Y)^{\mathrm{T}} \cdot (X,Y) - (m_x, m_y)^{\mathrm{T}}(m_x, m_y)\right\} =$

$= \begin{Vmatrix} DX & \mathrm{cov}(X,Y) \\ \mathrm{cov}(Y,X) & DY \end{Vmatrix} = \begin{Vmatrix} K_{XX} & K_{XY} \\ K_{YX} & K_{YY} \end{Vmatrix}$ - ifoda bilan aiqlanadi.

Kovariatsiyaning xossalari:

1. $K_{XY} = K_{YX}$;
2. Agar $X \perp Y$ bo‘lsa, u holda $K_{XY} = 0$;
3. Agar X va Y ixtiyoriy t.m.lar bo‘lsa, u holda $D(X \pm Y) = DX + DY \pm 2K_{XY}$;
4. $K_{CX,Y} = CK_{XY} = K_{X,CY}$ yoki $\mathrm{cov}(CX,Y) = C\,\mathrm{cov}(X,Y) = \mathrm{cov}(X,CY)$;
5. $K_{X+C,Y} = K_{XY} = K_{X,Y+C} = K_{X+C,Y+C}$ yoki $\mathrm{cov}(X+C,Y) = \mathrm{cov}(X,Y) = \mathrm{cov}(X,Y+C) = \mathrm{cov}(X+C,Y+C)$;
6. $|K_{XY}| \le \sigma_X \cdot \sigma_Y$.

Isboti. 1. (3.7.3) dan kelib chiqadi.

2. Agar $X \perp Y$ bo‘lsa, u holda $X - m_x$ va $Y - m_y$ lar ham bog‘liqsiz bo‘ladi va matematik kutilmaning xossasiga ko‘ra $K_{XY} = 0$.

3. $D(X \pm Y) = M((X \pm Y) - M(X \pm Y))^2 = M((X - MX) \pm (Y - MY))^2 =$

$= M(X - MX)^2 \pm 2M(X - MX)(Y - MY) + M(Y - MY)^2 = DX + DY \pm 2K_{XY}.$

4. $K_{CX,Y} = M(CX - MCX)(Y - MY) = M\left[C(X - MX)(Y - MY)\right] = CK_{XY}.$

5. $K_{X+C,Y} = M((X+C) - M(X+C))(Y - MY) = M(X + C - MX - C)(Y - MY) =$

$= M(X - MX)(Y - MY) = K_{XY}$

6. 3-xossani $\dfrac{X - m_x}{\sigma_x}$ va $\dfrac{Y - m_y}{\sigma_y}$ t.m.larga qo‘llasak,

$D\left(\dfrac{X - m_x}{\sigma_x} \pm \dfrac{Y - m_y}{\sigma_y}\right) = D\left(\dfrac{X - m_x}{\sigma_x}\right) + D\left(\dfrac{Y - m_y}{\sigma_y}\right) \pm$

$$\pm 2M\left[\left(\frac{X-m_x}{\sigma_x}-M\left(\frac{X-m_x}{\sigma_x}\right)\right)\left(\frac{Y-m_y}{\sigma_y}-M\left(\frac{Y-m_y}{\sigma_y}\right)\right)\right]=$$

$$=1+1\pm 2M\left(\frac{X-m_x}{\sigma_x}\cdot\frac{Y-m_y}{\sigma_y}\right)=2\left(1\pm\frac{K_{XY}}{\sigma_X\sigma_Y}\right).$$

Dispersiya manfiy bo‘lmasligidan $2\left(1\pm\frac{K_{XY}}{\sigma_X\sigma_Y}\right)\geq 0$, ya’ni $|K_{XY}|\leq \sigma_X\cdot\sigma_Y$.■

3-xossaga ko‘ra, agar $K_{XY}\neq 0$ bo‘lsa, X va Y t.m.lar bo‘gliq bo‘ladi. Bu holda X va Y t.m.lar korrelatsiyalangan deyiladi. Lekin $K_{XY}=0$ ekanligidan X va Y t.m.larning bog‘liqsizligi kelib chiqmaydi. Demak, X va Y t.m.larning bog‘liqsizligida ularning korrelatsiyalanmaganligi kelib chiqadi, teskarisi esa har doim ham o‘rinli emas.

✓ X va Y t.m.larning *korrelatsiya koeffitsienti*

$$r_{XY}=\frac{K_{XY}}{\sigma_X\sigma_Y}=\frac{\operatorname{cov}(X,Y)}{\sqrt{DX}\sqrt{DY}} \tag{3.7.7}$$

formula bilan aniqlanadi.

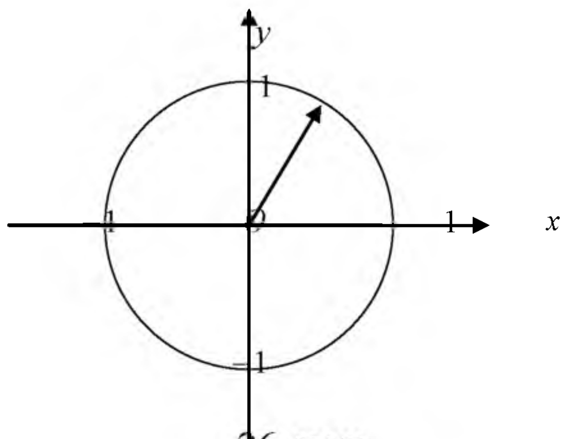
Korrelyatsiya koeffisiyentining xossalari:

1. $|r_{XY}|\leq 1$, ya’ni $-1\leq r_{XY}\leq 1$;
2. Agar $X\perp Y$ bo‘lsa, u holda $r_{XY}=0$;
3. Agar $|r_{XY}|=1$ bo‘lsa, u holda X va Y t.m.lar chiziqli funksional bog‘liq bo‘ladi, teskarisi ham o‘rinli.

Shunday qilib, bogliqsiz t.m.lar uchun $r_{XY}=0$, chiziqli bog‘langan t.m.lar uchun $|r_{XY}|=1$, qolgan hollarda $-1<r_{XY}<1$. Agar $r_{XY}>0$ bo‘lsa, t.m.lar musbat korrelatsiyalangan va aksincha agar $r_{XY}<0$ bo‘lsa, ular manfiy korrelyatsialangan deyiladi.

3.8 Ba’zi muhim ikki o‘lchovlik taqsimotlar

Doiradagi tekis taqsimot. Radiusi $R=1$ bo‘lgan doirada (X,Y) t.m. tekis taqsimotga ega bo‘lsin(26-rasm).



26-rasm.

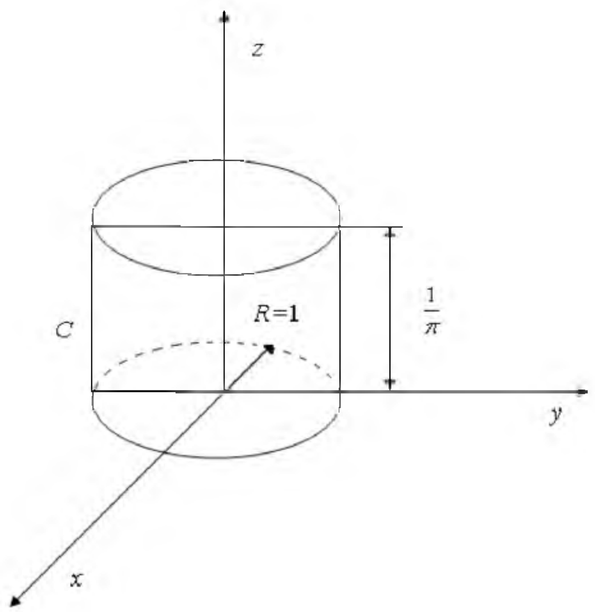
Demak, (X,Y) ning birgalikdagi zichlik funksiyasi

$$f(x,y)=\begin{cases} C, \text{agar } x^2+y^2\le 1, \\ 0, \text{agar } x^2+y^2>1. \end{cases}$$

O‘zgarmas C ni

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)dxdy=1 \text{, ya'ni } \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} C\,dxdy=1$$

shartdan aniqlaymiz. Bu karrali integralni geometrik ma'nosidan kelib chiqqan holda hisoblash osonroq(27-rasm).



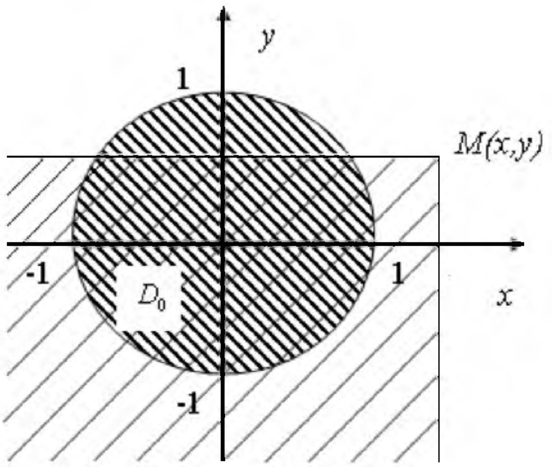
27-rasm.

$f(x,y)$ sirt va OXY tekislik bilan chegaralangan jismning hajmi 1 ga tengdir. Bizning holda bu asosi $\pi R^2 = \pi \cdot 1^2 = \pi$ va balandligi C bo‘lgan silindr hajmidir $V = \pi C = 1$. Demak, $C = \frac{1}{\pi}$ va izlanayotgan zichlik funksiyasi

$$f(x,y) = \begin{cases} \frac{1}{\pi}, \text{ agar } x^2 + y^2 \le 1, \\ 0, \text{ agar } x^2 + y^2 \le 1. \end{cases}$$

Unga mos taqsimot funksiyani hisoblaymiz:

$$F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v)\,dudv = \int_{-1}^{x} \int_{-\sqrt{1-u^2}}^{y} \frac{1}{\pi} dudv .$$



28-rasm.

Tabiiyki, bu integral $x^2 + y^2 \le 1$ doira bilan uchi M nuqtada bo‘lgan $D = \{(a,b) \in R^2 : a \le x, b \le y\}$- kvadrantning $\frac{1}{\pi}$ aniqligida kesishishidan hosil bo‘lgan soha D_0 yuzasiga tengdir(28-rasm). Tabiiyki, $x \le -1, -\infty < y < +\infty$ da $F(x,y) = 0$, chunki bu holda $D_0 = \varnothing$, endi $x > 1$ va $y > 1$ da $F(x,y) = 1$, chunki bu holda D_0 - soha $x^2 + y^2 \le 1$ doira bilan ustma-ust tushadi.

Endi X va Y larning marginal taqsimot funksuyalari F_X va F_Y larni hisoblaymiz: $-1 < x < 1$ da

$$F_X(x)=\int_{-\infty}^{x}\int_{-\infty}^{+\infty} f(u,v)\,dudv=\int_{-1}^{x}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}\frac{1}{\pi}dudv=\frac{1}{\pi}\cdot\int_{-1}^{x}\left(v\Big|_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}\right)du=\frac{1}{\pi}\cdot\int_{-1}^{x}2\sqrt{1-u^2}\,du=$$

$$=\frac{1}{\pi}\cdot\left[\left(x\sqrt{1-x^2}\right)+\arcsin u\Big|_{-1}^{x}\right]=\frac{1}{2}+\frac{1}{\pi}\cdot\left(x\sqrt{1-x^2}+\arcsin x\right).$$

Demak,

$$F_X(x)=\begin{cases}0, & \text{agar } x\le -1,\\ \frac{1}{2}+\frac{1}{\pi}\cdot\left(x\sqrt{1-x^2}+\arcsin x\right), & \text{agar } -1<x\le 1,\\ 1, & \text{agar } x>1.\end{cases}$$

Aynan shunga o‘xshash

$$F_Y(y)=\begin{cases}0, & \text{agar } y\le -1,\\ \frac{1}{2}+\frac{1}{\pi}\cdot\left(y\sqrt{1-y^2}+\arcsin y\right), & \text{agar } -1<y\le 1,\\ 1, & \text{agar } y>1.\end{cases}$$

Nihoyat, X va Y larning marginal zichliklarini hisoblaymiz:

$$f_X(x)=\int_{-\infty}^{+\infty} f(x,y)dy=\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\frac{1}{\pi}dy=\frac{2}{\pi}\cdot\sqrt{1-x^2},\quad -1\le x\le 1,$$

va shu kabi

$$f_Y(y)=\int_{-\infty}^{+\infty} f(x,y)dx=\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\frac{1}{\pi}dx=\frac{2}{\pi}\cdot\sqrt{1-y^2},\quad -1\le y\le 1.$$

Ko‘rinib turibdiki, $f(x,y)\ne f_X(x)\cdot f_Y(y)$, demak, X va Y bog‘liq t.m.lar ekan.

Shuni ta’kidlab o‘tish lozimki, tekis taqsimotga ega bo‘lgan har qanday (X,Y) juftlik doimo bog‘liq bo‘ladi deb aytish noto‘g‘ridir. Chunki X va Y larning bog‘liqlik xossalari ular qanday sohada tekis taqsimotga ega ekanligiga bog‘liqdir. Shu boisdan keyingi taqsimotni ko‘rib o‘tamiz.

Kvadratdagi tekis taqsimot. (X,Y) juftlik $[0,1]\times[0,1]$ kvadratda tekis taqsimotga ega bo‘lsin. U holda ular birgalikdagi taqsomot funksiyasi ko‘rinishi quyidagidek bo‘ladi:

$$F(x,y)=\begin{cases}0,\ x,y\le 0,\\ x\cdot y,\ 0<x,y<1,\\ 1,\ x,y\ge 1.\end{cases}$$

Bundan

$$F_X(x)=F(x,+\infty)=F(x,1)=\begin{cases}0,\ x\le 0,\\ x,\ 0<x<1,\\ 1,\ x\ge 1.\end{cases}$$

$$F_Y(y)=F(+\infty,y)=F(1,y)=\begin{cases}0,\ y\le 0,\\ y,\ 0<y<1,\\ 1,\ y\ge 1.\end{cases}$$

Demak, barcha $x,y\in R^1$ lar uchun $F(x,y)=F_X(x)\cdot F_Y(y)$, ya’ni X va Y bog‘liq emas ekan.

Ikki o‘lchovlik normal(Gauss) taqsimoti. (X,Y) tasodifiy vektor ikki o‘lchovli normal taqsimotga ega bo‘lsin. U holda (X,Y) ning birgalikdagi zichlik funksiyasi

$$f(x,y)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}}\cdot\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-a_1)^2}{\sigma_1^2}-2r\cdot\frac{(x-a_1)}{\sigma_1}\cdot\frac{(y-a_2)}{\sigma_2}+\frac{(y-a_2)^2}{\sigma_2^2}\right]\right\}.$$

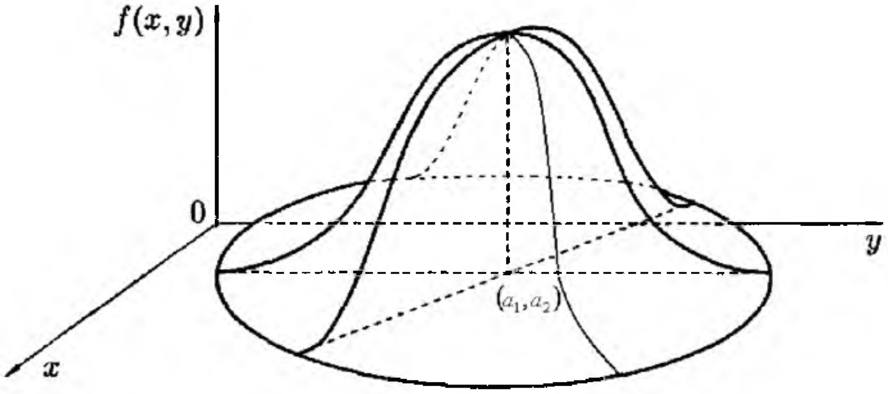
Geometrik nuqtayi nazardan $f(x,y)$ grafigi cho‘qqisi (a_1,a_2) nuqtada joylashgan «tog‘» shaklini bildiradi(29-rasm). Agarda biz bu tog‘ni OXY tekisligiga пarallel tekislik bilan kesadigan bo‘lsak, u holda kesilish chiziqlari quyidagi ellipslardan iborat bo‘ladi:

$\frac{(x-a_1)^2}{\sigma_1^2}-2r\cdot\frac{(x-a_1)}{\sigma_1}\cdot\frac{(y-a_2)}{\sigma_2}+\frac{(y-a_2)^2}{\sigma_2^2}=C$ -konstanta, bu yerda $a_1=MX$, $a_2=MY$, $\sigma_1^2=DX$, $\sigma_2^2=DY$, va $r=r_{X,Y}$ -korrelatsiya koeffitsientidir. Agar $r=0$ bo‘lsa, bu chiziqlar aylanalardan iborat bo‘lib qoladi. Biz r ning aynan korrelatsiya koeffisienti bo‘lishiga ishonch hosil qilish maqsadida

$$Z_1 = \frac{X - a_1}{\sigma_1^{\ 2}} \text{ va } Z_2 = \frac{Y - a_2}{\sigma_2^{\ 2}}$$

yangi t.m.larni kiritamiz. Tabiiyki, $MZ_k = 0, DZ_k = 1, k = 1,2$. U holda (Z_1, Z_2) ning zichlik funksiyasi

$$g(z_1, z_2) = \frac{1}{2\pi\sqrt{1-r^2}} \cdot \exp\left\{-\frac{z_1^{\ 2} - 2r \cdot z_1 z_2 + z_2^{\ 2}}{2(1-r^2)}\right\}.$$



29-rasm.

Endi korrelatsiya koeffitsientini hisoblaymiz:

$$r_{X,Y} = Cov(Z_1, Z_2) = MZ_1 Z_2 = \frac{1}{2\pi\sqrt{1-r^2}} \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z_1 z_2 g(z_1, z_2)\, dz_1 dz_2 =$$

$$= \frac{1}{2\pi\sqrt{1-r^2}} \int_{-\infty}^{\infty} z_2 e^{-\frac{z_2^2}{2}} \cdot \left(\int_{-\infty}^{\infty} (z_1 - rz_2 + rz_2) \cdot \exp\left\{-\frac{(z_1 - rz_2)^2}{2(1-r^2)}\right\} dz_1 \right) dz_2 =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z_2 \cdot e^{-\frac{z_2^2}{2}} \cdot \left[\frac{1}{\sqrt{2\pi} \cdot \sqrt{1-r^2}} \cdot \int_{-\infty}^{\infty} (z_1 - rz_2) \cdot \exp\left\{-\frac{(z_1 - rz_2)^2}{2(1-r^2)}\right\} \cdot d(z_1 - rz_2) \right] dz_2 +$$

$$+ \frac{r}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} z_2^{\ 2} \cdot e^{-\frac{z_2^2}{2}} \cdot \left[\frac{1}{\sqrt{2\pi} \cdot \sqrt{1-r^2}} \cdot \int_{-\infty}^{\infty} \exp\left\{-\frac{(z_1 - rz_2)^2}{2(1-r^2)}\right\} d(z_1 - rz_2) \right] dz_2 = r.$$

Oxirgi tenglikni hosil qilishda quyidagi integrallardan foydalandik:

$$\frac{1}{\sqrt{2\pi}\cdot\sqrt{1-r^2}}\cdot\int_{-\infty}^{\infty} u\cdot e^{-\frac{u^2}{2(1-r^2)}}du=0,\quad u=z_1-rz_2,$$

-markazlashtirilgan normal t.m.ning matematik kutilmasi;

$$\frac{1}{\sqrt{2\pi}\cdot\sqrt{1-r^2}}\cdot\int_{-\infty}^{\infty} e^{-\frac{u^2}{2(1-r^2)}}du=1,\quad u=z_1-rz_2,$$

-zichlik funksiya integrali;

$$\frac{1}{\sqrt{2\pi}}\cdot\int_{-\infty}^{\infty} z_2^{\,2}\cdot e^{-\frac{z_2^{\,2}}{2}}dz_2=1,$$

-standart normal t.m. dispersiyasi.

Demak, $r(X,Y)=r$ ekan. Agar ikki normal taqsimotga ega bo‘lgan X va Y t.m.lar bog‘liq bo‘lmasa, $r=0$ bo‘lishi r ning xossasidan kelib chiqadi. Endi shu t.m.lar uchun $r=0$ bo‘lsin. U holda

$$f(x,y)=\frac{1}{2\pi\sigma_1\sigma_2}\cdot\exp\left\{-\frac{1}{2}\left[\frac{(x-a_1)^2}{\sigma_1^{\,2}}+\frac{(y-a_2)^2}{\sigma_2^{\,2}}\right]\right\}=f_X(x)\cdot f_Y(y),$$

bu yerda

$$f_X(x)=\frac{1}{\sqrt{2\pi}\sigma_1}\cdot\exp\left\{-\frac{(x-a_1)^2}{2\sigma_1^{\,2}}\right\},\quad f_Y(y)=\frac{1}{\sqrt{2\pi}\sigma_2}\cdot\exp\left\{-\frac{(y-a_2)^2}{2\sigma_2^{\,2}}\right\}$$

funksiyalar $N(a_1,\sigma_1^2), N(a_2,\sigma_2^2)$ normal t.m.lar zichlik funksiyalaridir. Demak, t.m.lar korrelyatsiyalanmaganligidan ularning bog‘liqsizligi ham kelib chiqar ekan. Bu hol ikki o‘lchovlik normal taqsimotni boshqa taqsimotlardan ajratib turadi.

3.9 Xarakteristik funksiyalar va uning xossalari

Taqsimot funksiya bilan bir qatorda u haqidagi hamma ma’lumotni o‘z ichiga oluvchi xarakteristik funksiyalardan ham foydalaniladi. Xarakteristik funksiya yordamida bog‘liqsiz t.m.larning yig‘indisining taqsimotini topish, sonli xarakteristikalarni hisoblash bir muncha osonlashadi.

✓ X t.m.ning *xarakteristik funksiyasi* e^{itX} t.m.ning matematik kutilmasi bo‘lib, uni $\varphi_X(t)$ yoki $\varphi(t)$ orqali belgilaymiz. Shunday qilib, ta’rifga ko‘ra:

$$\varphi(t) = Me^{itX}. \tag{3.9.1}$$

Agar X t.m. $x_1, x_2, \ldots x_n, \ldots$ qiymatlarni $p_k = P\{X = x_k\}, k = 1, 2, \ldots$ ehtimolliklar bilan qabul qiluvchi diskret t.m. bo‘lsa, u holda uning xarakteristik funksiyasi

$$\varphi(t) = \sum_{k=1}^{\infty} e^{itx_k} p_k \tag{3.9.2}$$

formula orqali, agar zichlik funksiyasi $f(x)$ bo‘lgan uzluksiz t.m. bo‘lsa, u holda uning xarakteristik funksiyasi

$$\varphi(t) = \int_{-\infty}^{+\infty} e^{itx} f(x)dx \tag{3.9.3}$$

formula orqali aniqlanadi.

Xarakteristik funksiyaning xossalari:

1. Barcha $t \in R$ uchun quyidagi tengsizlik o‘rinli:

$$|\varphi(t)| \le \varphi(0) = 1.$$

2. Agar $Y = aX + b$ bo‘lsa, bu yerda a va b o‘zgarmas sonlar, u holda

$$\varphi_Y(t) = e^{itb}\varphi_X(at).$$

3. Agar X va Y t.m.lar bog‘liqsiz bo‘lsa, u holda $X+Y$ yig‘indining xarakteristik funksiyasi X va Y t.m.larning xarakteristik funksiyalari ko‘paytmasiga teng:

$$\varphi_{X+Y}(t) = \varphi_X(t) \cdot \varphi_Y(t).$$

4. Agar X t.m.ning k-tartibli boshlang'ich momenti $\alpha_k = MX^k$ mavjud bo'lsa, u holda unga mos xarakteristik funksiyaning k-tartibli hosilasi mavjud bo'lib, uning t=0 dagi qiymati

$$\varphi_X^{(k)}(0) = i^k M(X^k) = i^k \cdot \alpha_k .$$

Isboti. 1. $|\varphi(t)| = |Me^{itX}| \le M|e^{itX}| = M1 = 1$, chunki
$|e^{itX}| = |\cos tX + i\sin tX| = \sqrt{\cos^2 tX + \sin^2 tX} = 1$. $\varphi(0) = Me^0 = M1 = 1$.

2. $\varphi_Y(t) = Me^{itY} = Me^{it(aX+b)} = M(e^{itb}e^{iaXt}) = e^{itb}Me^{iatX} = e^{itb}\varphi_X(at)$.

3. $\varphi_{X+Y}(t) = Me^{it(X+Y)} = M(e^{itX}e^{itY}) = Me^{itX} \cdot Me^{itY} = \varphi_X(t) \cdot \varphi_Y(t)$. Bu xossa n ta bog'liqsiz tasodifiy miqdorlar yig'indisi uchun ham o'rinlidir.

4. Hisoblashdan ko'rinadiki, $\varphi_X^{(k)}(t) = \dfrac{d^k Me^{itX}}{dt^k} = i^k M(X^k e^{itX})$. Demak t=0 bo'lsa, $\varphi_X^{(k)}(0) = i^k M(X^k) = i^k \cdot \alpha_k$. ■

4-xossadan $\alpha_k = i^{-k}\varphi_X^{(k)}(0)$.

$$\alpha_1 = MX = -i\varphi'(0);\ \alpha_2 = MX^2 = -\varphi''(0);$$
$$DX = \alpha_2 - \alpha_1^2 = -\varphi''(0) + (\varphi'(0))^2. \tag{3.9.4}$$

3.7-misol. Agar $X \sim Bi(n;p)$ bo'lsa, u holda X t.m.ning xarakteristik funksiyasi, matematik kutilmasi va dispersiyasini toping.

X t.m. 0,1,2,…,n qiymatlarni $p_k = P\{X = k\} = C_n^k p^k q^{n-k}$ $k = 0,1,...,n$ ehtimolliklar bilan qabul qiladi. (3.9.2) va Nyuton binomi formulalaridan foydalansak, $\varphi(t) = \sum_{k=0}^{n} e^{itk}C_n^k p^k q^{n-k} = \sum_{k=0}^{n} C_n^k (e^{it} \cdot p)^k q^{n-k} = (e^{it}p + q)^n$, ya'ni X t.m.ning xarakteristik funksiyasi $\varphi(t) = (e^{it}p + q)^n$ ifoda bilan aniqlanishiga ishonch hosil qilamiz. (3.9.4) formulaga ko'ra: $MX = -i(n(e^{it}p + q)^{n-1} \cdot pe^{it} \cdot i)|_{t=0} = np$ va shu kabi $DX = npq$.

3.8-misol. Agar $X \sim N(a,\sigma)$ bo'lsa, u holda X ning xarakteristik funksiyasi, matematik kutilmasi va dispersiyasini toping.

(3.9.3) formulaga asosan: $\varphi(t) = \dfrac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty} e^{itx}e^{-\frac{(x-a)^2}{2\sigma^2}}dx =$

$$=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty} e^{-\frac{x^2-2(a+it\sigma^2)x+a^2}{2\sigma^2}}dx=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty} e^{-\frac{x^2-2x(a+it\sigma^2)+(a+it\sigma^2)^2+a^2-(a+it\sigma^2)^2}{2\sigma^2}}dx=$$

$$=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty} e^{-\frac{(x-(a+it\sigma^2))^2}{2\sigma^2}}\cdot e^{\frac{2ait\sigma^2+(it\sigma^2)^2}{2\sigma^2}}dx=\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{2ait\sigma^2-t^2\sigma^4}{2\sigma^2}}\int_{-\infty}^{+\infty} e^{-\frac{(x-(a+it\sigma^2))^2}{2\sigma^2}}dx=$$

$$=\frac{1}{\sqrt{2\pi}\sigma}e^{iat-\frac{t^2\sigma^2}{2}}\int_{-\infty}^{+\infty} e^{-\left(\frac{x-(a+it\sigma^2)}{\sqrt{2}\sigma}\right)^2}d\left(\frac{x-(a+it\sigma^2)}{\sqrt{2}\sigma}\right)\cdot\sqrt{2}\sigma=\frac{1}{\sqrt{\pi}}e^{iat-\frac{t^2\sigma^2}{2}}\sqrt{\pi}=e^{iat-\frac{t^2\sigma^2}{2}}$$

$\left(\int_{-\infty}^{+\infty} e^{-u^2}du=\sqrt{\pi}\text{ Puasson integrali}\right)$. Shunday qilib, agar $X\sim N(a,\sigma)$ bo‘lsa, u holda $\varphi(t)=e^{iat-\frac{t^2\sigma^2}{2}}$. Endi X t.m.ning matematik kutilmasi va dispersiyasini hisoblaymiz.

$$MX=\left[-i\varphi'(0)\right]=-ie^{iat-\frac{t^2\sigma^2}{2}}(ia-t\sigma^2)\big|_{t=o}=-i\cdot 1\cdot ia=a,$$

$$DX=\left[-\varphi''(0)-(\varphi'(0))^2\right]=-\left(-\sigma^2e^{iat-\frac{t^2\sigma^2}{2}}+(ia-t\sigma^2)^2e^{iat-\frac{t^2\sigma^2}{2}}\right)\Big|_{t=o}+(ia)^2=$$

$$=\sigma^2-i^2a^2+i^2a^2=\sigma^2.$$

III bobga doir misollar

1. (X,Y) ikki o‘lchovli uzluksiz t.m.ning birgalikdagi zichlik funksiyasi $f(x,y)=\frac{C}{(1+x^2)(1+y^2)}$ ko‘rinishida berilgan bo‘lsa, quyidagilarni toping:

1) o‘zgarmas son C; 2) $F(x,y)$; 3) $P\{X<1, Y<1\}$; 4) $f(x)$ va $f(y)$.

2. Agar (X,Y) vektor taqsimoti quyidagicha bo‘lsa:

X \ Y	-1	0	1
0	0.1	0.2	0.1
1	0.2	0.3	0.1

$Z=XY$ ning matematik kutilmasini hisoblang.

3. (X,Y) ikki o‘lchovlik uzluksiz t.m. uchlari $O(0,0)$, $A(0,4)$, $B(4,0)$ nuqtalarda bo‘lgan uchburchak ichida tekic taqsimlangan(ya’ni $f(x,y)=c$). Quyidagilarni hisoblang: 1) birgalikdagi zichlik funksiyasi $f(x,y)$; 2) $f(x)$va $f(y)$; 3) $A=\{0<X<1,\ 1<Y<3\}$ hodisaning ehtimolligini.

4. (X,Y) tasodifiy vektor zichligi $f(x,y)=\begin{cases} x+y, & 0\le x\le 1,\ 0\le y\le 1, \\ 0, & \text{aks holda} \end{cases}$ bo‘lsa, MX va MY larni hisoblang.

5. Agar (X,Y) tasodifiy vektorning taqsimoti

Y \ X	0	1
0	1/8	0
1	1/4	1/8
2	1/8	3/8

bo‘lsa, u holda $M(X+Y)=MX+MY$, $D(X+Y)=DX+DY+2Cov(x,y)$ tengliklar o‘rinli ekanligini ko‘rsating.

6. Quyida (X,Y) ikki o‘lchovli uzluksiz t.m.ning birgalikdagi zichlik funksiyasi berilgan: $f(x,y)=\begin{cases} Cxy, & \text{agar } (x,y)\in D, \\ 0, & \text{agar } (x,y)\notin D \end{cases}$, bu erda D tekislikdagi quyidagi shartlarni qanoatlantiruvchi soha: $\begin{cases} y>-x, \\ y<2, \\ x<0. \end{cases}$ O‘zgarmas son C ni toping, X va Y t.m.lar bog‘liq ekanligini ko‘rsating.

7. Agar $X\sim Bi(2;0.2), Y\sim Bi(1;0.8)$ va $X\perp Y$ bo‘lsa, u holda $Z=X+Y$ t.m.ning taqsimot funksiyasini toping va $F_Z(1)$ ni hisoblang.

8. Agar X va Y t.m.larning birgalikdagi taqsimoti

X \ Y	-1	0	1	2
-1	0.05	0.3	0.15	0.05
1	0.1	0.05	0.25	0.05

bo‘lsa, u holda $Z=|Y-X|$ va $U=Y^2-X^2$ larning taqsimotlarini toping.

9. (X,Y) ikki o‘lchovlik diskret t.m.ning birgalikdagi taqsimot jadvali berilgan:

X \ Y	1	2	3	4
1	0.07	0.04	0.11	0.11
2	0.08	0.11	0.06	0.08
3	0.09	0.13	0.10	0.02

X va Y t.m.lar bog‘liq yoki bog‘liqsizligini tekshiring va cov(X,Y) ni hisoblang.

10. X va Y t.m.larning birgalikdagi zichlik funksiyasi berilgan:

$$f(x,y)=\begin{cases} a(1-xy^3), & \text{agar } |x|\le 1, |y|\le 1, \\ 0, & \text{aks holda.} \end{cases}$$

O‘zgarmas son a va korrelatsiya koeffitsientini hisoblang.

11. X va Y t.m.larning birgalikdagi zichlik funksiyasi berilgan:

$$f(x,y)=\begin{cases} C(x+y), & \text{agar } 0\le x\le 1, 0\le y\le 1, \\ 0, & \text{aks holda.} \end{cases}$$

1) X va Y t.m.lar bog‘liqmi? 2) X va Y t.m.larning matematik kutilmasi va dispersiyasini hisoblang.

12. (X,Y) tasodifiy vektorning birgalikdagi taqsimoti

$$F(x,y)=\begin{cases} 0, & x<0, y<0, \\ \frac{xy}{8}, & 0\le x\le 2,\ 0\le y\le 4, \\ 1, & x>2, y>2, \end{cases}$$

bo‘lsa, X va Y o‘zaro bog‘liqmi?

13. Agar (X,Y) tasodifiy vektorning birgalikdagi taqsimoti berilgan bo‘lsa, $Cov(X,Y)$ ni hisoblang.

X \ Y	1	2
1	$\frac{1}{3}$	$\frac{1}{3}$
2	0	$\frac{1}{3}$

14. $X \sim R(-a,a)$ bo‘lsa, X va $Y=X^2$ lar uchun $Cov(X,Y)$ ni hisoblang. X va Y lar bog‘liqmi?

15. Agar X va Y bog‘liqsiz, bir xil taqsimlangan va $MX^2,\ MY^2<\infty$ bo‘lsa, u holda $Cov(X+Y,X-Y)=0$ ekanini isbotlang.

16. Agar $X\sim E(1)$, $DY=2$ va $D(X-Y)=3$ bo‘lsa, r_{XY} ni hisoblang.

17. Agar $\begin{cases} X: & -\frac{\pi}{2} & 0 & \frac{\pi}{2} \\ P_X: & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{cases}$ bo‘lsa, $Y=\sin X$ va $Z=\cos X$ uchun $Cov(Y,Z)=0$, ammo Y va Z bog‘liqligini ko‘rsating.

18. Agar (X,Y) zichlik funksiyasi $f(x,y)=e^{-x-y}$, $x,y\ge 0$ bo‘lsa, shartli zichlik funksiyalar $f(x/Y=y)$ va $g(y/X=x)$ ni hisoblang.

19. Agar (X,Y) tasodifiy vektorning birgalikdagi zichlik funksiyasi $f(x,y)=\begin{cases} x+2y, & (x,y)\in[0,1]\times[0,1], \\ 0, & \text{aks holda} \end{cases}$ bo‘lsa, $M(Y/X=x)$ ni $x=1$ da hisoblang.

20. Agar (X,Y) ning birgalikdagi taqsimoti bo‘lsa,

X \ Y	1	2	3
1	2/9	1/9	0
2	1/9	0	1/9
3	2/9	1/9	1/9

r_{XY} ni hisoblang .

21. Agar *X* va *Y* t.m.larning birgalikdagi zichlik funksiyasi $f(x,y)=\frac{1}{3\pi}e^{-\frac{x^2+4y^2}{6}}$ bo‘lsa, u holda (*X*,*Y*) tasodifiy nuqtaning $\{|x|\le 1, |y|\le 2\}$ sohaga tushishi ehtimolligini toping.

22. [*a*,*b*] oraliqda tekis taqsimlangan *X* t.m.ning xarakteristik funksiyasini toping.

23. Agar *X* t.m. *a* parametrli Puasson taqsimotiga ega bo‘lsa, uning matematik kutilmasini xarakteristik funksiya yordamida hisoblang.

24. X t.m.ning zichlik funksiyasi berilgan: $f(x)=\begin{cases} -2x, & \text{agar } x\in[-1,0], \\ 0, & \text{agar } x\notin[-1,0], \end{cases}$ $\varphi_X(t)$ ni hisoblang.

25. Agar *X* va *Y* t.m.larning birgalikdagi taqsimoti quyidagi jadval yordamida berilgan bo‘lsa, *P*(*X*=1/*Y*=1), *P*(*X*=0/*Y*=1), *MX* va *MY* larni toping.

Y \ X	0	1	2
0	1/4	1/8	1/8
1	1/8	1/8	1/4

IV bob. Tasodifiy miqdorlarning funksiyalari

4.1 Bir argumentning funksiyalari

✓ Agar X t.m.ning har bir qiymatiga biror qoida bo‘yicha mos ravishda Y t.m.ning bitta qiymati mos qo‘yilsa, u holda Y ni X tasodifiy argumentning funksiyasi deyiladi va $Y=\varphi(X)$ kabi yoziladi.

X diskret t.m. $x_1,x_2,...,x_n$ qiymatlarni mos $p_1,p_2,...,p_n$ ehtimolliklar bilan qabul qilsin: $p_i=P\{X=x_i\},\ i=1,2,...,n$. Ravshanki, $Y=\varphi(X)$ t.m. ham diskret t.m. bo‘ladi va uning qabul qiladigan qiymatlari $y_1=\varphi(x_1)$, $y_2=\varphi(x_2),\dots,y_n=\varphi(x_n)$, mos ehtimolliklari esa $p_1,p_2,...,p_n$ bo‘ladi. Demak, $p_i=P\{Y=y_i\}=P\{Y=\varphi(x_i)\},\ i=1,2,...,n$. Shuni ta’kidlash lozimki, X t.m.ning har xil qiymatlariga mos Y t.m.ning bir xil qiymatlari mos kelishi mumkin. Bunday hollarda qaytarilayotgan qiymatlarning ehtimolliklarini qo‘shish kerak bo‘ladi.

$Y=\varphi(X)$ t.m.ning matematik kutilmasi va dispersiyasi quyidagi tengliklar orqali aniqlanadi:

$$MY=\sum_{i=1}^{n}\varphi(x_i)p_i,\quad DY=\sum_{i=1}^{n}(\varphi(x_i)-MY)^2p_i\ .$$

4.1-misol. X diskret t.m.ning taqsimot jadvali berilgan:

X	-1	1	2
p	0.1	0.2	0.6

Agar: 1) $Y=X^2$; 2) $Y=2X+10$ bo‘lsa, MY ni hisoblang.

1) Y t.m.ning qabul qiladigan qiymatlari: $y_1=\varphi(x_1)=(-1)^2=1,\quad y_2=1^2=1,$ $y_3=2^2=4$, ya’ni uning qabul qiladigan qiymatlai 1 va 4. Y t.m. X t.m.ning -1 va 1 qiymatlarida 1 qiymat qabul qilganligi uchun

$p_1=P\{Y=1\}=P\{X=-1\}+P\{X=1\}=0.1+0.3=0.4$,

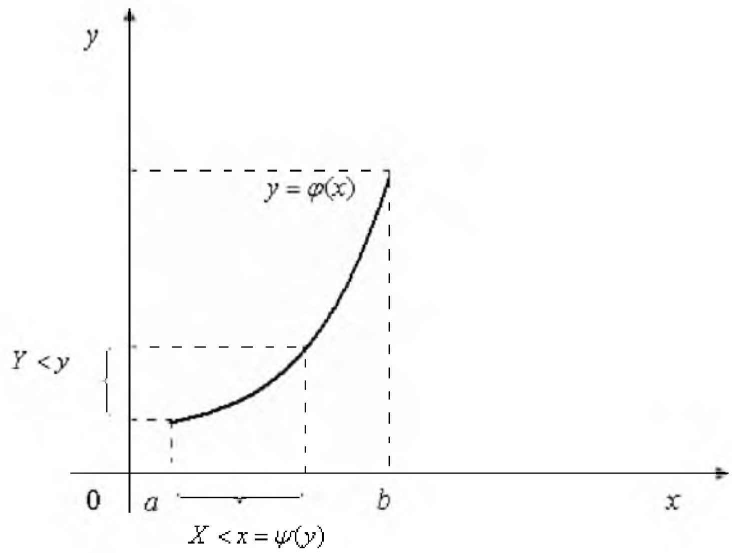
$p_2=P\{Y=4\}=P\{X=2\}=0.6$. Demak, $\begin{cases}Y: & 1,\ \ 4\\ P: & 0.4,\ \ 0.6\end{cases}$ va

$MY=1\cdot0.4+4\cdot0.6=2.8$.

2) Y t.m.ning taqsimot qonuni quyidagi ko‘rinishga ega: $\begin{cases}Y: & 8,\ \ 12,\ \ 14\\ P: & 0.1,\ \ 0.3,\ \ 0.6\end{cases}$.

$MY=8\cdot0.1+12\cdot0.3+14\cdot0.6=12.8$.

Zichlik funksiyasi $f(x)$ bo'lgan X uzluksiz t.m. berilgan bo'lsin. Y t.m. esa X t.m.ning funksiyasi $Y = \varphi(X)$. Y t.m.ning taqsimotini topamiz. $Y = \varphi(X)$ funksiya X t.m.ning barcha qiymatlarida uzluksiz, (a,b) intervalda qat'iy o'suvchi va differensiallanuvchi bo'lsin, u holda $y = \varphi(x)$ funksiyaga teskari $x = \psi(y)$ funksiya mavjud. Y t.m.ning taqsimot funksiyasi $G(y) = P\{Y < y\}$ formula orqali aniqlanadi. $\{Y < y\}$ hodisa $\{X < \psi(y)\}$ hodisaga ekvivalent (30-rasm).



30-rasm.

Yuqoridagilarni e'tiborga olsak,

$$G(y) = P\{Y < y\} = P\{X < \psi(y)\} = F_X(\psi(y)) = \int_a^{\psi(y)} f(x)dx . \qquad (4.1.1)$$

(4.1.1) ni y bo'yicha differensiallaymiz va Y t.m.ning zichlik funksiyasini topamiz: $g(y) = \frac{G(y)}{dy} = f(\psi(y)) \cdot \frac{d}{dy}(\psi(y)) = f(\psi(y))\psi'(y)$.

Demak,

$$g(y) = f(\psi(y))\psi'(y) . \qquad (4.1.2)$$

Agar $y = \varphi(x)$ funksiya (a,b) intervalda qat'iy kamayuvchi bo'lsa, u holda $\{Y < y\}$ hodisa $\{X < \psi(y)\}$ hodisaga ekvivalent. Shuning uchun,

$$G(y)=\int_{\psi(y)}^{b} f(x)dx=-\int_{b}^{\psi(y)} f(x)dx.$$

Bu yerdan,

$$g(y)=-f(\psi(y))\psi'(y) \qquad (4.1.3)$$

Zichlik funksiya manfiy bo‘lmasligini hisobga olib, (4.1.2) va (4.1.3) formulalarni umumlashtirish mumkin:

$$g(y)=f(\psi(y))\left|\psi'(y)\right|. \qquad (4.1.4)$$

Agar $y=\varphi(x)$ funksiya (a,b) intervalda monoton bo‘lmasa, u holda $g(y)$ ni topish uchun (a,b) intervalni n ta monotonlik bo‘lakchalarga ajratish, har biri bo‘yicha teskari funksiyasi ψ_i ni topish va quyidagi formuladan foydalanish kerak:

$$g(y)=\sum_{i=1}^{n} f(\psi_i(y))\left|\psi_i'(y)\right|. \qquad (4.1.5)$$

Agar X zichlik funksiyasi $f(x)$ bo‘lgan uzluksiz t.m. bo‘lsa, u holda $Y=\varphi(X)$ t.m.ning sonli xarakteristikalarini hisoblash uchun Y t.m.ning taqsimotini qo‘llash shart emas:

$$MY=M(\varphi(X))=\int_{-\infty}^{+\infty}\varphi(x)f(x)dx,$$

$$DY=D(\varphi(X))=\int_{-\infty}^{+\infty}(\varphi(x)-MY)^2 f(x)dx. \qquad (4.1.6)$$

4.2-misol. X zichlik funksiyasi $f(x)$ bo‘lgan uzluksiz t.m. bo‘lsa, Y=-5X+2 t.m.ning zichlik funksiyasini toping.

$y=-5x+2$ funksiya $(-\infty;+\infty)$ intervalda monoton kamayuvchi. Teskari funksiyasi $x=\frac{1}{5}(2-y)=\psi(y)$ mavvud, $\psi'(y)=-\frac{1}{5}$. U holda (4.1.4) formulaga ko‘ra, $g(y)=f\left(\frac{2-y}{5}\right)\cdot\left|-\frac{1}{5}\right|=\frac{1}{5}f\left(\frac{2-y}{5}\right), y\in(-\infty;+\infty)$.

4.2-misol yordamida taqsimot va zichlik funksiyalarning formulalarini tekshiramiz:

$$G(y)=P\{Y<y\}=P\{-5X+2<y\}=P\left\{X>\frac{2-y}{5}\right\}=$$

$$=1-P\left\{X\le\frac{2-y}{5}\right\}=1-\left(P\left\{X<\frac{2-y}{5}\right\}+P\left\{X=\frac{2-y}{5}\right\}\right)=$$

$$=1-P\left\{X<\frac{2-y}{5}\right\}=1-F_X\left(\frac{2-y}{5}\right).$$

Demak, $G(y)=1-F_X\left(\frac{2-y}{5}\right)$, u holda $g(y)=G'(y)=\left(1-F_X\left(\frac{2-y}{5}\right)\right)'_y=$

$=-f_X\left(\frac{2-y}{5}\right)\cdot\left(\frac{2-y}{5}\right)'_y=-f\left(\frac{2-y}{5}\right)\cdot\left(-\frac{1}{5}\right)$, ya'ni

$g(y)=\frac{1}{5}f\left(\frac{2-y}{5}\right)$, $y\in(-\infty;+\infty)$.

Y=aX+b chiziqli almashtirish taqsimot xarakterini o‘zgartirmaydi: normal t.m.dan normal t.m.; tekis t.m.dan tekis t.m. hosil bo‘ladi.

4.3-misol. X t.m. $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ intervalda tekis taqsimlangan. $Y=\cos X$ t.m.ning matematik kutilmasini a) $g(y)$ zichlik funksiyani topib; b) $g(y)$ zichlik funksiyani topmasdan hisoblang.

a) X t.m.ning zichlik funksiyasi $f(x)=\begin{cases}\frac{1}{\pi}, x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\\ 0, x\notin\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\end{cases}$ bo‘ladi. $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ intervalda $y=\cos x$ funksiya monoton emas: $x\in\left(-\frac{\pi}{2},0\right)$ intervalda o‘suvchi, $x\in\left(0,\frac{\pi}{2}\right)$ intervalda esa kamayuvchi. Birinchi intervalda teskari funksiya, $x_1=-\arccos y=\psi_1(y)$ ikkinchi intervalda esa $x_2=\arccos y=\psi_2(y)$ ga teng. U holda (4.1.5) formulaga asosan

$$g(y)=f(\psi_1(y))\left|\psi_1'(y)\right|+f(\psi_2(y))\left|\psi_2'(y)\right|=\frac{1}{\pi}\left|\frac{1}{\sqrt{1-y^2}}\right|+\frac{1}{\pi}\left|-\frac{1}{\sqrt{1-y^2}}\right|=\frac{2}{\pi\sqrt{1-y^2}}$$

Demak,

$$g(y)=\begin{cases}\dfrac{2}{\pi\sqrt{1-y^2}}, & \text{agar } 0<y<1,\\ o, & \text{agar } y\le 0 \text{ yoki } y\ge 1.\end{cases}$$

U holda

$$MY=\left[\int_{-\infty}^{+\infty} yg(y)dy\right]=\int_0^1 y\cdot\frac{2}{\pi\sqrt{1-y^2}}dy=$$

$$=-\frac{2}{\pi}\cdot\frac{1}{2}\int_0^1(1-y^2)^{-\frac{1}{2}}d(1-y^2)=-\frac{1}{\pi}\cdot 2\sqrt{1-y^2}\,\Big|_0^1=\frac{2}{\pi}.$$

b) (4.1.6) formuladan foydalanamiz:

$$MY=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos x\cdot\frac{1}{\pi}dx=\frac{1}{\pi}\sin x\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\frac{1}{\pi}(1-(-1))=\frac{2}{\pi}.$$

4.2 Ikki argumentning funksiyalari

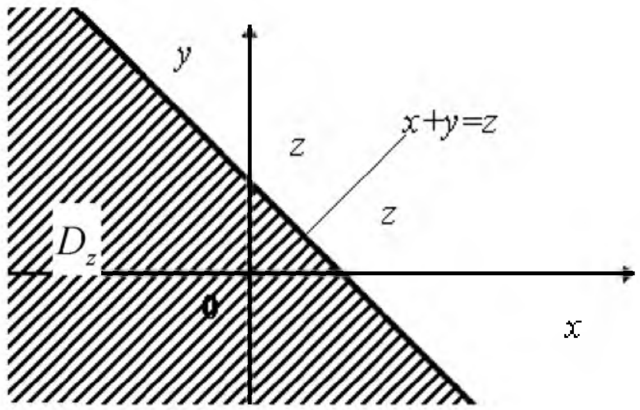
✓ Agar X va Y t.m.lar qabul qiladigan qiymatlarining har bir juftligiga biror qoidaga ko‘ra Z t.m. mos qo‘yilsa, u holda Z t.m. X va Y ikki tasodifiy argumentning funksiyasi deyiladi va $Z=\varphi(X,Y)$ kabi belgilanadi.

$Z=\varphi(X,Y)$ funksiyaning amaliyotda muhim ahamiyatga ega bo‘lgan xususiy holi $Z=X+Y$ t.m.ning taqsimotini topamiz.

(X,Y) ikki o‘lchovli uzluksiz t.m. $f(X,Y)$ birgalikdagi zichlik funksiyaga ega bo‘lsin. (3.4.3) formuladan foydalanib, $Z=X+Y$ t.m.ning taqsimot funksiyasini topamiz:

$$F_Z(z)=P\{Z<z\}=P\{X+Y<z\}=\iint_{D_z} f(x,y)dxdy, \tag{4.2.1}$$

bu yerda $D_z=\{(x,y):x+y<z\}$ (31-rasm).



31-rasm.

U holda $F_Z(z)=\int\limits_{-\infty}^{+\infty}\left(\int\limits_{-\infty}^{z-x} f(x,y)dy\right)dx$. Hosil bo‘lgan tenglikni z o‘zgaruvchi bo‘yicha differensiallab, $Z=X+Y$ t.m. uchun zichlik funksiyaga ega bo‘lamiz:

$$f_Z(z)=\int\limits_{-\infty}^{+\infty} f(x,z-x)dx\,. \qquad (4.2.2)$$

Agar X va Y t.m.lar bog‘liqsiz bo‘lsa, $f(x,y)=f(x)\cdot f(y)$ tenglik o‘rinli bo‘ladi va (4.2.2) formula

$$f_Z(z)=f_{X+Y}(z)=\int\limits_{-\infty}^{+\infty} f_1(x)f_2(z-x)dx \qquad (4.2.3)$$

ko‘rinishda bo‘ladi.

✓ Bog‘liqsiz t.m.lar yig‘indisining taqsimoti shu t.m.lar taqsimotlarining *kompozitsiyasi* deyiladi. Z t.m.ning zichlik funksiyasi $f_{X+Y}=f_X * f_Y$ ko‘rinishda yoziladi, bu yerda * - kompozitsiya belgisi.

Xuddi shunday agar $Z=Y+X$ ko‘rinishda yozib olsak, $f_Z(z)$ uchun boshqa formulaga ega bo‘lamiz:

$$f_Z(z)=\int\limits_{-\infty}^{+\infty} f(z-y,y)dy\,,$$

agar X va Y t.m.lar bog‘liqsiz bo‘lsa, u holda

$$f_Z(z)=\int_{-\infty}^{+\infty} f_1(z-y)f_2(y)dy\,.$$

$Z=X-Y, Z=X\cdot Y$ t.m.larning taqsimotlarini topish ham xuddi shunga o‘xshash amalga oshiriladi.

4.4-misol. Agar X va Y t.m.lar bog‘liqsiz bo‘lib, $X\sim N(0,1)$, $Y\sim N(0,1)$ bo‘lsa, $Z=X+Y$ ning taqsimotini toping. (4.2.3) formulaga asosan:

$$f_Z(z)=\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{(z-x)^2}{2}}dx=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-\frac{2x^2-2zx+z^2}{2}}dx=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-\frac{2\left(\left(x-\frac{z}{2}\right)^2+\frac{z^2}{4}\right)}{2}}dx=$$

$$=\frac{1}{2\pi}e^{\frac{z^2}{4}}\int_{-\infty}^{+\infty}e^{-\left(x-\frac{z}{2}\right)^2}d\left(x-\frac{z}{2}\right)=\frac{1}{2\pi}e^{-\frac{z^2}{4}}\sqrt{\pi}=\frac{1}{\sqrt{2}\sqrt{2\pi}}e^{-\frac{z^2}{2(\sqrt{2})^2}},$$

ya’ni $f_{X+Y}(z)=\frac{1}{\sqrt{2}\sqrt{2\pi}}e^{-\frac{z^2}{2(\sqrt{2})^2}}$. Demak, bog‘liqsiz, normal taqsimlangan t.m.lar ($a=0,\sigma=1$ parametrli) yig‘indisi ham normal taqsimlangan ($a=0,\sigma=\sqrt{2}$ parametrli) bo‘lar ekan.

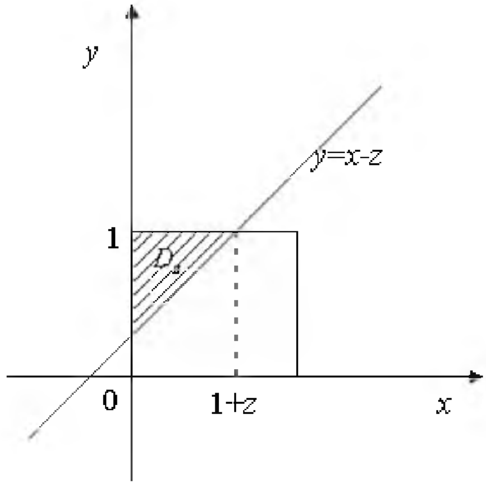
4.5-misol. X va Y t.m.larning birgalikdagi zichlik funksiyasi berilgan:

$$f(x,y)=\begin{cases}x+y, \text{ agar } 0\le x\le 1, 0\le y\le 1,\\ 0, \text{ aks holda.}\end{cases}$$

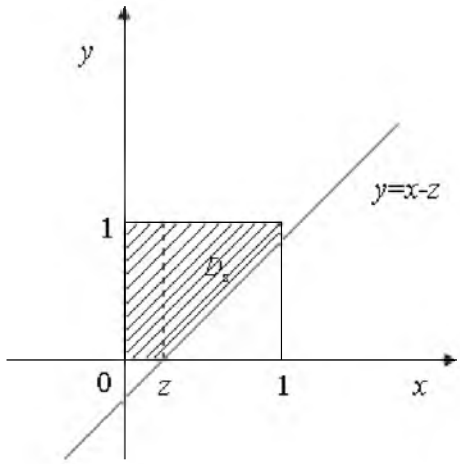
$Z=X-Y$ t.m.ning zichlik funksiyasini toping.
Avval Z t.m.ning taqsimot funksiyasi $F_Z(z)$ ni topamiz.

$$F_Z(z)=P\{Z<z\}=P\{X-Y<z\}=\iint_{D_z}(x+y)dxdy\,,$$

bu yerda $D_z = \{(x,y): x - y < z\}$, z ixtiyoriy son. 32–rasmda D_z sohani $-1 < z \le 0$ bo‘lgandagi integrallash sohasi, 33-rasmda esa $0 < z \le 1$ bo‘lgandagi integrallash sohasi tasvirlangan.



32-rasm.



33-rasm.

$-1 < z \le 0$ bo‘lganda:

$$F_Z(z) = \iint_{D_z} (x+y)dxdy = \int_0^{1+z} dx \int_{x-z}^{1} (x+y)dy = \int_0^{1+z} dx \left(xy + \frac{y^2}{2} \right) \Big|_{x-z}^{1} =$$

$$= \int_0^{1+z} \left(x + \frac{1}{2} - x^2 + xz - \frac{(x-z)^2}{2} \right) dx = \left(\frac{x^2}{2} + \frac{1}{2}x - \frac{x^3}{3} + z\frac{x^2}{2} - \frac{(x-z)^3}{6} \right) \Big|_0^{1+z} =$$

$$= \frac{(1+z)^2}{2} + \frac{1+z}{2} - \frac{(1+z)^3}{3} + \frac{z(1+z)^2}{2} - \frac{1}{6} - \frac{z^3}{6} = \frac{(1+z)^2}{2}.$$

Agar $0 < z \le 1$ bo‘lsa,

$$F_Z(z) = \iint_{D_z} (x+y)dxdy = \int_0^{z} dx \int_{x-z}^{1} (x+y)dy + \int_z^{1} dx \int_{x-z}^{1} (x+y)dy =$$

$$= \int_0^{z} dx \left(xy + \frac{y^2}{2} \right) \Big|_0^1 + \int_z^1 dx \left(xy + \frac{y^2}{2} \right) \Big|_{x-z}^{1} = \int_0^z \left(x + \frac{1}{2} \right) dx +$$

$$+\int_z^1\left(x+\frac{1}{2}-x^2+xz-\frac{(x-z)^2}{2}\right)dx=\left(\frac{x^2}{2}+\frac{x}{2}\right)\Big|_0^x+$$

$$+\left(\frac{x^2}{2}+\frac{x}{2}-\frac{x^3}{3}+z\frac{x^2}{2}-\frac{(x-z)^3}{6}\right)\Big|_z^1=\frac{-z^2+2z+1}{2}.$$

Yuqoridagi hisoblardan

$$F_z(z)=\begin{cases}0, & \text{agar } z\le -1,\\ \dfrac{(1+z)^2}{2}, & \text{agar -1<z}\le 0,\\ \dfrac{-z^2+2z+1}{2}, & \text{agar 0<z}\le 1,\\ 1, & \text{agar z>1.}\end{cases}$$

Zichlik funksiyasi esa,

$$F_z'(z)=f_Z(z)=\begin{cases}0, & \text{agar } z\le -1, z>1,\\ z+1, & \text{agar -1<z}\le 0,\\ 1-z, & \text{agar 0<z}\le 1.\end{cases}$$

IV bobga doir misollar

1. X diskret t.m.ning taqsimot jadvali berilgan:

X	-2	-1	0	1	2	3
p	0.10	0.20	0.30	0.25	0.10	0.05

a) $Y=2X^2-3$; b) $Y=\sqrt{X+2}$; c) $Y=\sin\frac{\pi}{3}X$ t.m.larning taqsimot qonunlarini toping.

2. Diskret X t.m.ning taqsimot qonuni

X	-2	-1	0	1	2
p	0.2	0.1	0.3	0.1	0.3

bo‘lsa, $Y=X^2+1$, $Z=|X|$ t.m.larning taqsimot qonunlarini toping.

3. Agar $X \sim R[-2,2]$ bo‘lsa, $Y = X + 1$ t.m.ning zichlik funksiyasi va dispersiyasini toping.

4. Agar $X \sim N(0,1)$ bo‘lsa, a) $Y = 3X^3$; b) $Y = |X|$ t.m.larning zichlik funksiyasini toping.

5. X∈R(0,2) va Y=-3X+1 bo‘lsa, Y t.m.ning taqsimot funksiyasini toping.

6. Taqsimoti

X	-1	0	1
P	0.4	0.1	0.5

bo‘lgan t.m.dan tuzilgan $У=2^X$ t.m.ning matematik kutilmasi va dispersiyasini toping.

7. Taqsimoti P(X=-1)=P(X=1)=1/2 bo‘lgan t.m.dan olingan $Z_1=\cos X\pi$, $Z_2=\sin X\pi$ t.m.larning matematik kutilmalari va dispersiyalarini toping.

8. Taqsimoti

X	-1	0	1	2
P	0.2	0.3	0.3	0.2

bo‘lgan t.m.dan tuzilgan У=|X| t.m. ning matematik kutilmasi va dispersiyasini toping.

9. X∈Bi(2,1/3); $\begin{cases} Y: & -1, \ 1 \\ P: & 1/4, \ 3/4 \end{cases}$ va X⊥Y bo‘lsa, Z=X+2Y t.m.ning matematik kutilmasi va dispersiyasini toping.

10. Ikkita tanga va kub tashlash tajribasida “gerb”lar soni X va kubdagi ochkolar soni У ning birgalikdagi taqsimot jadvalini tuzing va DX, DУ larni hisoblang.

11. X uzluksiz t.m.ning zichlik funksiyasi berilgan bo‘lsin: $f(x) = \begin{cases} e^{-x}, & \text{agar } x \geq 0, \\ 0, & \text{agar } x < 0. \end{cases}$ a) $Y = 2X - 1$; b) $Y = X^2$ t.m.larning zichlik funksiyalarini toping.

12. Agar $X \sim R[0,4]$, $Y \sim R[0,1]$ va $X \perp Y$ bo‘lsa, $Z = X + Y$ t.m.ning zichlik funksiyasini toping.

13. Bog‘liqsiz X va Y t.m.larning taqsimot qonunlari berilgan

X	-1	1	2
p	0.4	0.3	0.3

Y	-1	0	1	2
p	0.2	0.25	0.3	0.25

bo‘lsa, $X+Y$ va XY t.m.larning taqsimot qonunlarini toping.

V bob. Ehtimollar nazariyasining limit teoremalari

Ehtimollar nazariyasining limit teoremalari deb nomlanuvchi qator tasdiq va teoremalarni keltiramiz. Ular yetarlicha katta sondagi tajribalarda t.m.lar orasidagi bog‘lanishni ifodalaydi. Limit teoremalar shartli ravishda ikki guruhga bo‘linadi. Birinchi guruh teoremalar katta sonlar qonunlari(KSQ) deb nomlanadi. Ular o‘rta qiymatning turg‘unligini ifodalaydi: yetarlicha katta sondagi tajribalarda t.m.larning o‘rta qiymati tasodifiyligini yo‘qotadi. Ikkinchi guruh teoremalar markaziy limit teoremalar(MLT) deb nomlanadi. Yetarlicha katta sondagi tajribalarda t.m.lar yig‘indisining taqsimoti normal taqsimotga intilishi shartini ifodalaydi. KSQ ni keltirishdan avval yordamchi tengliklarni isbotlaymiz.

5.1 Chebishev tengsizligi

Teorema(Chebishev). Agar X t.m. DX dispersiyaga ega bo‘lsa, u holda $\forall\varepsilon>0$ uchun quyidagi tengsizlik o‘rinli:

$$P\{|X-MX|\geq\varepsilon\}\leq\frac{DX}{\varepsilon^2}. \qquad (5.1.1)$$

(5.1.1) tengsizlik Chebishev tengsizligi deyiladi.

Isboti. $P\{|X-a|\geq\varepsilon\}$ ehtimollik X t.m.ning $[a-\varepsilon;a+\varepsilon]$ oraliqqa tushmasligi ehtimolligini bildiradi bu yerda $a=MX$. U holda

$$P\{|X-a|\geq\varepsilon\}=\int_{-\infty}^{a-\varepsilon}dF(x)+\int_{a+\varepsilon}^{+\infty}dF(x)=\int_{|x-a|\geq\varepsilon}dF(x)=$$

$$=\int_{|x-a|\geq\varepsilon}1\cdot dF(x)\leq\int_{|x-a|\geq\varepsilon}\frac{(x-a)^2}{\varepsilon^2}dF(x),$$

chunki $|x-a|\geq\varepsilon$ integrallash sohasini $(x-a)^2\geq\varepsilon^2$ ko‘rinishda yozish mumkin. Bu yerdan $\frac{(x-a)^2}{\varepsilon^2}\geq1$ ekanligi kelib chiqadi. Agar integrallash sohasi kengaytirilsa, musbat funksiyaning integrali faqat kattalashishini hisobga olsak,

$$P\{|X-a|\geq\varepsilon\}\leq\frac{1}{\varepsilon^2}\int\limits_{|x-a|\geq\varepsilon}(x-a)^2dF(x)\leq\frac{1}{\varepsilon^2}\int\limits_{-\infty}^{+\infty}(x-a)^2dF(x)=\frac{1}{\varepsilon^2}DX\ . \qquad ■$$

Chebishev tengsizligini quyidagi ko'rinishda ham yozish mumkin:

$$P\{|X-MX|<\varepsilon\}\geq 1-\frac{DX}{\varepsilon^2}. \tag{5.1.2}$$

Chebishev tengsizligi ihtiyoriy t.m.lar uchun o'rinli. Xususan, X t.m. binomial qonun bo'yicha taqsimlangan bo'lsin, $P\{X=m\}=C_n^m p^m q^{n-m}$, $m=0,1,...,n,$ $q=1-p\in(0,1)$. U holda $MX=a=np,\ DX=npq$ va (5.1.1) dan

$$P\{|m-np|<\varepsilon\}\geq 1-\frac{npq}{\varepsilon^2}; \tag{5.1.3}$$

n ta bog'liqsiz tajribalarda ehtimolligi $p=M\left(\frac{m}{n}\right)=a$, dispersiyasi $D\left(\frac{m}{n}\right)=\frac{qp}{n}$ bo'lgan hodisaning $\frac{m}{n}$ chastotasi uchun,

$$P\left\{\left|\frac{m}{n}-p\right|<\varepsilon\right\}\geq 1-\frac{qp}{n\varepsilon^2}. \tag{5.1.4}$$

X t.m.ni $[\varepsilon;+\infty)$ oraliqga tushushi ehtimolligini baholashni Markov tengsizligi beradi.

Teorema(Markov). Manfiy bo'lmagan, matematik kutilmasi MX chekli bo'lgan X t.m. uchun $\forall\varepsilon>0$ da

$$P\{X\geq\varepsilon\}\leq\frac{MX}{\varepsilon} \tag{5.1.5}$$

tengsizlik o'rinli.

Isboti. Quyidagi munosabatlar o'rinlidir:

$$P\{X\geq\varepsilon\}=\int\limits_{\varepsilon}^{+\infty}dF(x)\leq\int\limits_{\varepsilon}^{+\infty}\frac{x}{\varepsilon}dF(x)=\frac{1}{\varepsilon}\int\limits_{0}^{+\infty}xdF(x)=\frac{MX}{\varepsilon}\ . \qquad ■$$

(5.1.5) tengsizlikdan (5.1.1) ni osongina keltirib chiqarish mumkin.

(5.1.5) tengsizlikni quyidagi ko‘rinishda ham yozish mumkin:

$$P\{X<\varepsilon\}\geq 1-\frac{MX}{\varepsilon}. \qquad (5.1.6)$$

5.1.-misol. X diskret t.m.ning taqsimot qonuni berilgan:

$\begin{cases} X: & 1 \quad 2 \quad 3 \\ P_X: & 0.3 \quad 0.2 \quad 0.5. \end{cases}$ Chebishev tengsizligidan foydalanib, $P\{|X-MX|<\sqrt{0.4}\}$ ehtimollikni baholaymiz. X t.m.ning sonli xarakteristikalarini hisoblaymiz: $MX=1\cdot 0.3+2\cdot 0.2+3\cdot 0.5=2.2$; $DX=1^2\cdot 0.3+2^2\cdot 0.2+3^2\cdot 0.5-2.2^2=0.76$.

Chebishev tengsizligiga ko‘ra: $P\{|X-2.2|<\sqrt{0.4}\}\geq 1-\frac{0.76}{0.4}=0.9.$

5.2 Katta sonlar qonuni Chebishev va Bernulli teoremalari

Ehtimollar nazariyasi va uning tadbiqlarida ko‘pincha yetarlicha katta sondagi t.m.lar yig‘indisi bilan ish ko‘rishga to‘g‘ri keladi. Yig‘indidagi har bir t.m.ning tajriba natijasida qanday qiymatni qabul qilishini oldindan aytib bo‘lmaydi. Shuning uchun katta sondagi t.m.lar yig‘indisining taqsimot qonunini hisoblash burmuncha qiyinchilik tug‘diradi. Lekin ma’lum shartlar ostida yetarlicha katta sondagi t.m.lar yig‘indisi tasodifiylik xarakterini yo‘qotib borar ekan. Amaliyotda juda ko‘p tasodifiy sabablarning birgalikdagi ta’siri tasodifga deyarli bog‘liq bo‘lmaydigan natijaga olib keladigan shartlarni bilish juda muhimdir. Bu shartlar “Katta sonlar qonuni” deb ataluvchi teoremalarda keltiriladi. Bular qatoriga Chebishev va Bernulli teoremalari kiradi.

✓ $X_1, X_2, ... X_n, ...$ t.m.lar o‘zgarmas son A ga ehtimollik bo‘yicha yaqinlashadi deyiladi, agar $\forall\varepsilon>0$ uchun

$$\lim_{n\to\infty} P\{|X_n-A|<\varepsilon\}=1$$

munosabat o‘rinli bo‘lsa. Ehtimollik bo‘yicha yaqinlashish $X_n \xrightarrow[n\to\infty]{P} A$ kabi belgilanadi.

✓ $X_1, X_2, ... X_n, ...$ t.m.lar ketma-ketligi mos ravishda $MX_1, MX_2, ... MX_n, ...$ matematik kutilmalarga ega bo‘lib, $\forall\varepsilon > 0$ son uchun $n \to \infty$ da

$$\lim_{n\to\infty} P\left\{\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\sum_{i=1}^{n} MX_i\right| < \varepsilon\right\} = 1$$

munosabat bajarilsa, $X_1, X_2, ... X_n$ t.m.lar ketma-ketligi *katta sonlar qoniniga bo‘ysunadi* deyiladi.

Teorema(Chebishev). Agar bog‘liqsiz $X_1, X_2, ... X_n, ...$ t.m.lar ketma-ketligi uchun shunday $\exists C > 0$ bo‘lib $DX_i \le C,\ i = 1, 2, ...$ tengsizliklar o‘rinli bo‘lsa, u holda $\forall\varepsilon > 0$ uchun

$$\lim_{n\to\infty} P\left\{\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\sum_{i=1}^{n} MX_i\right| < \varepsilon\right\} = 1 \qquad (5.2.1)$$

munosabat o‘rinli bo‘ladi.

Isboti. $DX_i \le C,\ i = 1, 2, ...$ bo‘lgani uchun

$$D\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} D\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} DX_i = \frac{1}{n^2}\left(DX_1 + ... + DX_n\right) \le \frac{1}{n^2}\left(C + ... + C\right) =$$

$= \frac{1}{n^2} Cn = \frac{C}{n}$. U holda Chebishev tengsizligiga ko‘ra:

$$P\left\{\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\sum_{i=1}^{n} MX_i\right| < \varepsilon\right\} \ge 1 - \frac{D\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)}{\varepsilon^2} \ge 1 - \frac{C}{n\varepsilon^2}. \qquad (5.2.2)$$

Endi $n \to \infty$ da limitga o‘tsak, $\lim_{n\to\infty} P\left\{\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\sum_{i=1}^{n} MX_i\right| < \varepsilon\right\} = 1$. ■

Natija. Agar $X_1, X_2, ... X_n, ...$ bog‘liqsiz va bir xil taqsimlangan t.m.lar va $MX_i = a,\ DX_i = \sigma^2$ bo‘lsa, u holda $\forall\varepsilon > 0$ uchun quyidagi munosabat o‘rinli

$$\lim_{n\to\infty} P\left\{\left|\frac{1}{n}\sum_{i=1}^{n} X_i - a\right| < \varepsilon\right\} = 1. \qquad (5.2.3)$$

Bernulli teoremasi katta sonlar qonuninig sodda shakli hisoblanadi. U nisbiy chastotaning turg‘unligini asoslaydi.

Teorema(Bernulli). Agar A hodisaning bitta tajribada ro‘y berishi ehtimolligi p bo‘lib, n ta bog‘liqsiz tajribada bu hodisa n_A marta ro‘y bersa, u holda $\forall\varepsilon>0$ uchun

$$\lim_{n\to\infty}P\left\{\left|\frac{n_A}{n}-p\right|<\varepsilon\right\}=1 \qquad (5.2.4)$$

munosabat o‘rinli.

Isboti. $X_1, X_2, ...X_n$ indikator t.m.larni quyidagicha kiritamiz: agar *i*-tajribada A hodisa ro‘y bersa, $X_i=1$; agar ro‘y bermasa $X_i=0$. U holda n_A ni quyidagi ko‘rinishda yozish mumkin: $n_A=\sum_{i=1}^{n}X_i$. X_i t.m.ning taqsimot qonuni ixtiyoriy i da: $\begin{cases} X_i: 0 & 1 \\ P: 1-p & p \end{cases}$ bo‘ladi. X_i t.m.ning matematik kutilmasi $MX_i=1\cdot p+0\cdot(1-p)=p$ ga, dispersiyasi $DX_i=(0-p)^2(1-p)+(1-p)^2p= \; =p(1-p)=pq$. X_i t.m.lar bog‘liqsiz va ularning dispersiyalari chegaralangan, $p(1-p)=p-p^2=\frac{1}{4}-\left(p-\frac{1}{2}\right)^2\le\frac{1}{4}$.

U holda Chebishev teoremasiga asosan: $\lim_{n\to\infty}P\left\{\left|\frac{1}{n}\sum_{i=1}^{n}X_i-\frac{1}{n}\sum_{i=1}^{n}MX_i\right|<\varepsilon\right\}=1$

va $\frac{1}{n}\sum_{i=1}^{n}X_i=\frac{n_A}{n}$; $\frac{1}{n}\sum_{i=1}^{n}MX_i=\frac{1}{n}np=p$ bo‘lgani uchun $\lim_{n\to\infty}P\left\{\left|\frac{n_A}{n}-p\right|<\varepsilon\right\}=1$.

■

5.3 Markaziy limit teorema

Markaziy limit teorema t.m.lar yig‘indisi taqsimoti va uning limiti – normal taqsimot orasidagi bog‘lanishni ifodalaydi. Bir xil taqsimlangan t.m.lar uchun markaziy limit teoremani keltiramiz.

Teorema. $X_1, X_2, ...X_n$ bog‘liqsiz, bir xil taqsimlangan, $MX_i=a$ chekli matematik kutilma va $DX_i=\sigma^2, i=\overline{1,n}$ dispersiyaga ega bo‘lsin,

$0<\sigma^2<\infty$ u holda $Z_n=\dfrac{\sum_{i=1}^{n}X_i-M\left(\sum_{i=1}^{n}X_i\right)}{\sqrt{D\left(\sum_{i=1}^{n}X_i\right)}}=\dfrac{\sum_{i=1}^{n}X_i-na}{\sigma\sqrt{n}}$ t.m.ning taqsimot qonuni $n\to\infty$ da standart normal taqsimotga intiladi

$$F_{Z_n}(x)=P\{Z_n<x\}\xrightarrow[n\to\infty]{}\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-t^2/2}dt. \qquad (5.3.1)$$

Demak, (5.3.1) ga ko‘ra yetarlicha katta *n* larda $Z_n\sim N(0,1)$, $S_n=X_1+\ldots+X_n$ yig‘indi esa quyidagi normal qonun bo‘yicha taqsimlangan bo‘ladi: $S_n\sim N(na,\sqrt{n}\sigma)$. Bu holda $\sum_{i=1}^{n}X_i$ t.m. asimptotik normal taqsimlangan deyiladi.

Agar X t.m. uchun $MX=0, DX=1$ bo‘lsa X t.m. markazlashtirilgan va normallashtirilgan(yoki standart) t.m. deyiladi. (5.3.1) formula yordamida yetarlicha katta *n* larda t.m.lar yig‘indisi bilan bog‘liq hodisalar ehtimolligini hisoblash mumkin. $S_n=\sum_{i=1}^{n}X_i$ t.m.ni standartlashtirsak, yetarlicha katta *n* larda

$$P\left\{\alpha\le\sum_{i=1}^{n}X_i\le\beta\right\}=P\left\{\frac{\alpha-na}{\sigma\sqrt{n}}\le\frac{\sum_{i=1}^{n}X_i-na}{\sigma\sqrt{n}}\le\frac{\beta-na}{\sigma\sqrt{n}}\right\}\approx\Phi\left(\frac{\beta-na}{\sigma\sqrt{n}}\right)-\Phi\left(\frac{\alpha-na}{\sigma\sqrt{n}}\right),$$

yoki

$$P\{\alpha\le S_n\le\beta\}\approx\Phi\left(\frac{\beta-MS_n}{\sqrt{DS_n}}\right)-\Phi\left(\frac{\alpha-MS_n}{\sqrt{DS_n}}\right). \qquad (5.3.2)$$

5.2-misol. X_i bog‘liqsiz t.m.lar [0,1] oraliqda tekis taqsimlangan bo‘lsa, $Y=\sum_{i=1}^{100}X_i$ t.m.ning taqsimot qonunini toping va $P\{55<Y<70\}$ ehtimollikni hisoblang.

Markaziy limit teorema shartlari bajarilganligi uchun, Y t.m.ning zichlik funksiyasi $f_Y(y) \approx \frac{1}{\sqrt{2\pi}\sigma_y} e^{-\frac{(y-MY)^2}{2\sigma_y^2}}$ bo‘ladi. Tekis taqsimot matematik kutilmasi va dispersiyasi formulasidan $MX_i = \frac{0+1}{2} = \frac{1}{2}$, $DX_i = \frac{(1-0)^2}{12} = \frac{1}{12}$ bo‘ladi. U holda $MY = M\left(\sum_{i=1}^{100} X_i\right) = \sum_{i=1}^{100} MX_i = 100 \cdot \frac{1}{2} = 50,$

$DY = D\left(\sum_{i=1}^{100} X_i\right) = \sum_{i=1}^{100} DX_i = 100 \cdot \frac{1}{12} = \frac{25}{3}$, $\sigma_Y = \frac{5\sqrt{3}}{3}$, shuning uchun, $f_Y(y) \approx \frac{3}{5\sqrt{6\pi}} e^{-\frac{3(y-50)^2}{50}}$. (5.3.2) formulaga ko‘ra,

$$P\{55 < S_n < 70\} \approx \Phi\left(\frac{70-50}{\frac{5\sqrt{3}}{3}}\right) - \Phi\left(\frac{55-50}{\frac{5\sqrt{3}}{3}}\right) = \Phi\left(4\sqrt{3}\right) - \Phi\left(\sqrt{3}\right) \approx 0.04.$$

V bobga doir misollar

1. Bog‘liqsiz bir xil taqsimlangan $X_1, X_2, ..., X_n, ...$ t.m.lar ketma-ketligining taqsimot qonuni berilgan

X_n	a	$-a$
P	$\frac{n}{2n+1}$	$\frac{n+1}{2n+1}$

Bu ketma-ketlik K.S.Q. bo‘ysunadimi?

2. Bog‘liqsiz bir xil taqsimlangan $X_1, X_2, ..., X_n, ...$ t.m.lar ketma-ketligining taqsimot qonuni berilgan

X_n	$-n\alpha$	0	$n\alpha$
P	$1/2^n$	$1-\frac{1}{2^n}$	$1/2^n$

Bu ketma-ketlik K.S.Q. bo‘ysunadimi?

3. Diskret t.m. taqsimot qonuni berilgan:

X	0.1	0.4	0.6
P	0.2	0.3	0.5

Chebishev tengsizligidan foydalanib $|X - MX| < \sqrt{0.4}$ ni baholang.

4. $X_1, X_2, ..., X_n, ...$ bog‘liqsiz t.m.lar ketma-ketligi quyidagi taqsimotga ega bo‘lsin: $P\{X_n = \pm 2^n\} = 2^{-(2n+1)}$, $P\{X_n = 0\} = 1 - 2^{-2n}$. Bu t.m.lar uchun K.S.Q. o‘rinlimi?

5. Detalning nostandart bo‘lish ehtimolligi 0.2 ga teng. 400 ta detaldan iborat partiyada nostandart detal chiqishning chastotasi va ehtimoli orasidagi farqning moduli 0.05 dan kichik bo‘lishini ehtimolligini baholang.

6. Chebishev tengsizligidan foydalanib, quyidagi ehtimollikni baholang: simmetrik tanga 500 marta tashlanganda gerb tushushlari soni k uchun $200 \le k \le 300$ o‘rinli.