CHAPTER 1

Introduction to Fourier Series

1.1. Introduction

Throughout this chapter, E denotes the collection of all functions $f : [-\pi, \pi] \to \mathbb{C}$ which are piecewise continuous on the interval $[-\pi, \pi]$. This means that any function $f \in E$ has at most a finite number of points of discontinuity, at each of which f need not be defined but must have one sided limits which are finite. We further adopt the convention that any two functions $f, g \in E$ are considered equal, denoted by $f = g$, if $f(x) = g(x)$ for every $x \in [-\pi, \pi]$ with at most a finite number of exceptions.

Suppose now that $f \in E$. The purpose of our study is to represent such a function f in the form

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{1.1}$$

where the coefficients $a_0, a_1, a_2, \ldots$ and $b_1, b_2, b_3, \ldots$ depend only on the function f. We also wish to represent f in the form

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{\mathrm{i}nx}, \tag{1.2}$$

where again the coefficients $\ldots, c_{-2}, c_{-1}, c_0, c_1, c_2, \ldots$ depend only on the function f.

A few questions arise immediately. The function f is piecewise continuous on $[-\pi, \pi]$, while each of the summands in the series in (1.1) and (1.2) is continuous on $[-\pi, \pi]$. Can the series, if convergent, represent the function f in a satisfactory way? Under what conditions are the series convergent? Do we need more terms in the series? How are the coefficients a_n, b_n and c_n calculated in terms of the given function f? How do we interpret such coefficients?

1.2. Some Examples of Real Fourier Series

Let us investigate Fourier series of the type (1.1), where the coefficients a_n and b_n are real.

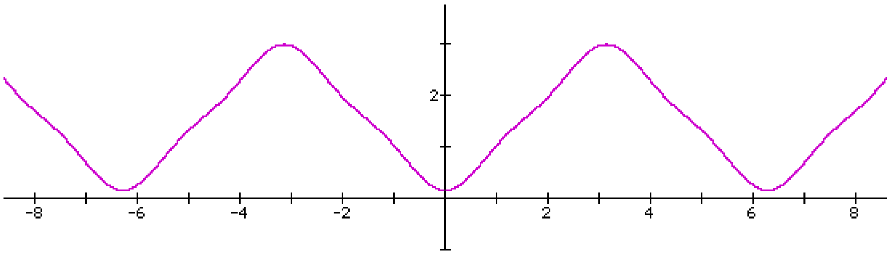
Example 1.2.1. Consider the function $f : [-\pi, \pi] \to \mathbb{C}$, given by $f(x) = |x|$ for every $x \in [-\pi, \pi]$. We shall show in Example 3.1.2 that this function has Fourier series

$$|x| \sim \frac{\pi}{2} - \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n^2} \cos nx,$$

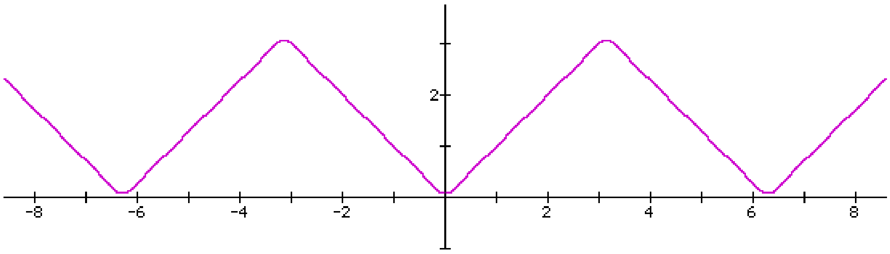
with partial sums

$$S_m(x) = \frac{\pi}{2} - \sum_{\substack{n=1 \\ n \text{ odd}}}^{m} \frac{4}{\pi n^2} \cos nx.$$

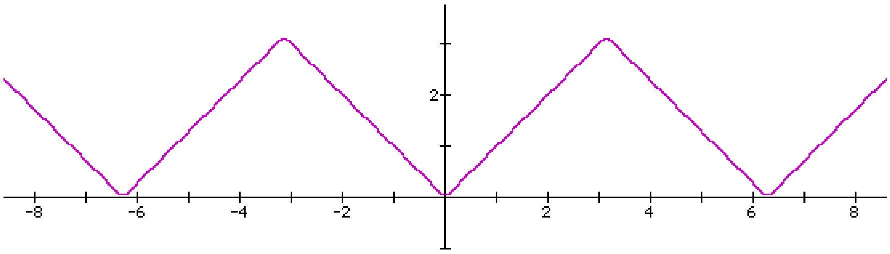
Clearly the Fourier series and all the partial sums are periodic functions with period 2π, so we extend the definition of f to the real line $\mathbb{R}$ by writing $f(x+2\pi) = f(x)$ for every $x \in \mathbb{R}$. The following graphs represent the partial sums $S_m(x)$ for $m = 3, 7, 11, 15$:



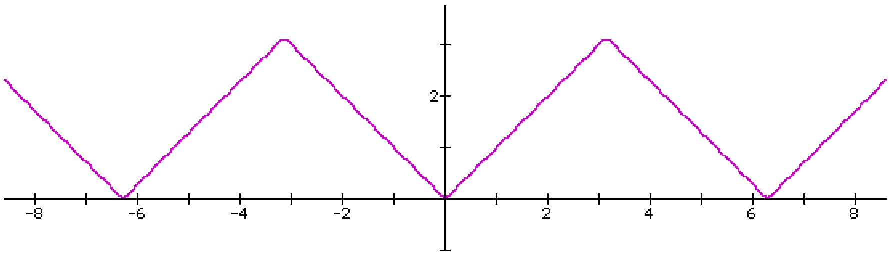
$m = 3$



$m = 7$



$m = 11$



$m = 15$

We see that as m increases, the graph for the partial sum $s_m(x)$ gets closer and closer to the graph for the 2π-periodic function f on $\mathbb{R}$. Note that this function f is continuous on $\mathbb{R}$, as are all the partial sums. In some sense, this example is not very interesting. We expect $S_m(x) \to |x|$ for every $x \in [-\pi, \pi]$. The Fourier series converges absolutely for every $x \in [-\pi, \pi]$, and uniformly in any subset of this.

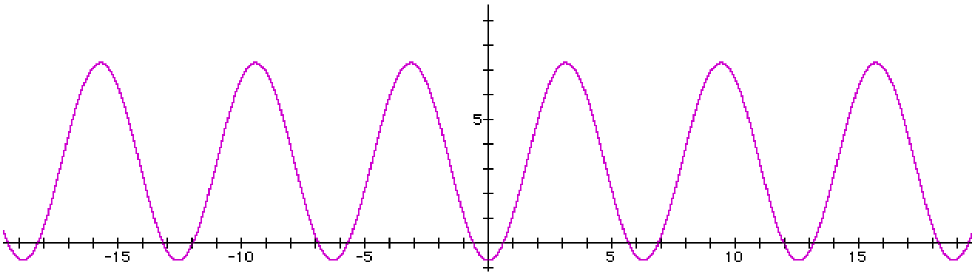
EXAMPLE 1.2.2. Consider the function $f : [-\pi, \pi] \to \mathbb{C}$, given by $f(x) = x^2$ for every $x \in [-\pi, \pi]$. We shall show in Example 3.1.4 that this function has Fourier series

$$x^2 \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx,$$

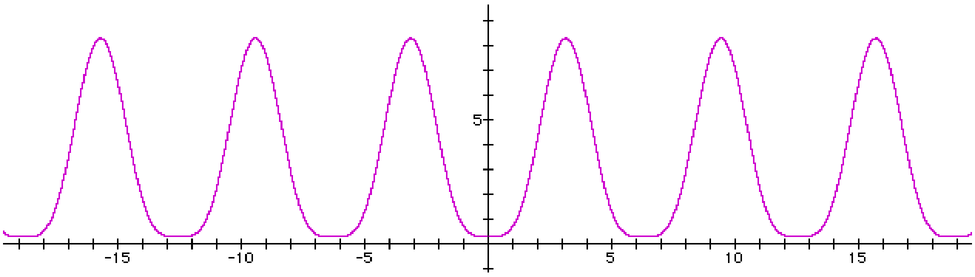
with partial sums

$$S_m(x) = \frac{\pi^2}{3} + \sum_{n=1}^{m} \frac{4(-1)^n}{n^2} \cos nx.$$

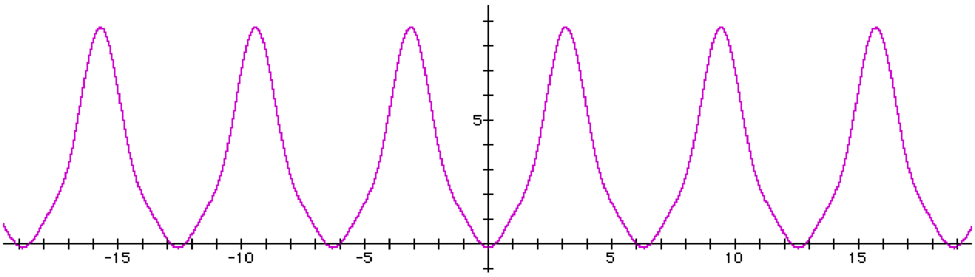
Again the Fourier series and all the partial sums are periodic functions with period 2π. As in the previous example, we extend the definition of f to the real line $\mathbb{R}$ by writing $f(x+2\pi) = f(x)$ for every $x \in \mathbb{R}$. The following graphs represent the partial sums $S_m(x)$ for $m = 1, 2, 3, 4, 5, 6$:



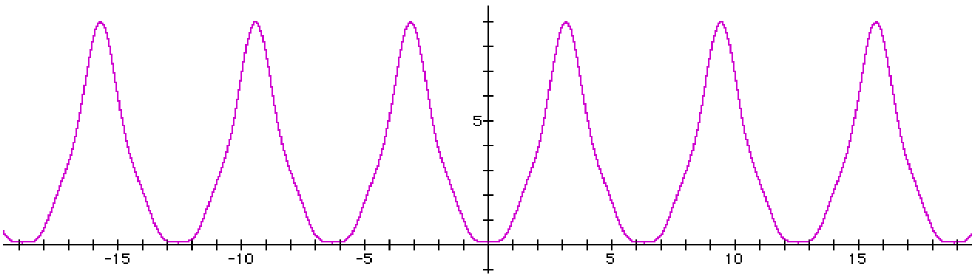
$m = 1$



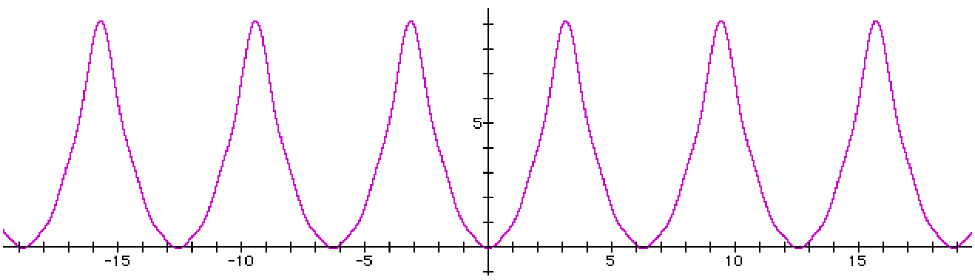
$m = 2$



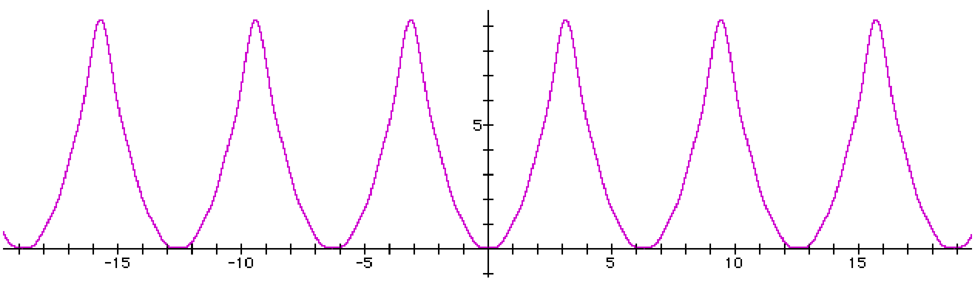
$m = 3$



$m = 4$



$m = 5$



$m = 6$

Notice the effect of the sign change $(-1)^n$ in the partial sum. The partial sum $s_1(x)$ is negative for some values of x, and this is over-corrected by the contribution from the term corresponding to $n = 2$. Then the term corresponding to $n = 3$ over-corrects this over-correction again, and so on. As m increases, the graph for the partial sum $s_m(x)$ gets closer and closer to the graph for the 2π-periodic function f on $\mathbb{R}$. Note that this function f is continuous on $\mathbb{R}$, as are all the partial sums. We expect $S_m(x) \to x^2$ for every $x \in [-\pi, \pi]$. The Fourier series converges absolutely for every $x \in [-\pi, \pi]$, and uniformly in any subset of this.

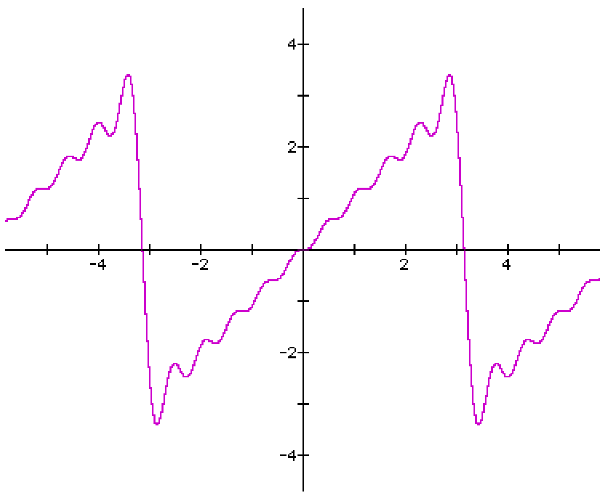
EXAMPLE 1.2.3. Consider the function $f : [-\pi, \pi] \to \mathbb{C}$, given by $f(x) = x$ for every $x \in [-\pi, \pi]$. We shall show in Example 3.1.1 that this function has Fourier series

$$x \sim \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx,$$

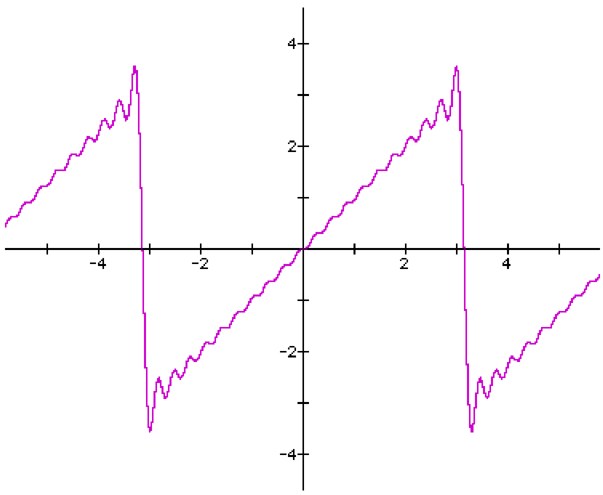
with partial sums

$$S_m(x) = \sum_{n=1}^{m} \frac{2(-1)^{n+1}}{n} \sin nx.$$

Again the Fourier series and all the partial sums are periodic functions with period 2π. By altering our definition of f at one or both of the points $x = \pm\pi$, we may extend f to a periodic function with period 2π the real line $\mathbb{R}$ by writing $f(x + 2\pi) = f(x)$ for every $x \in \mathbb{R}$. The following graph represents the partial sums $S_m(x)$ for $m = 10, 20$:

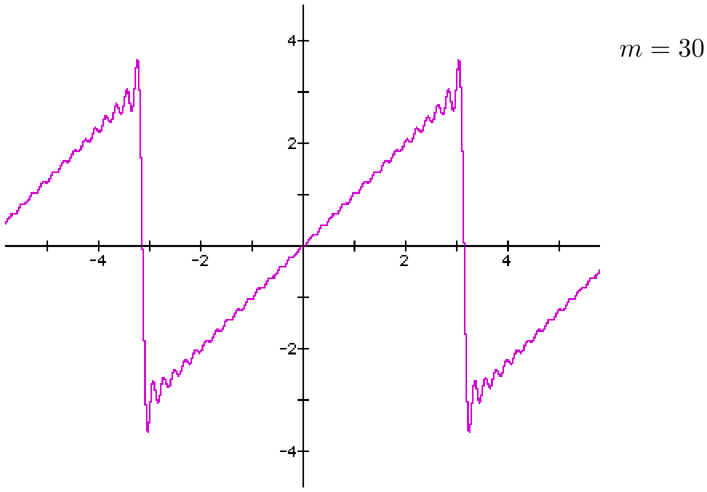


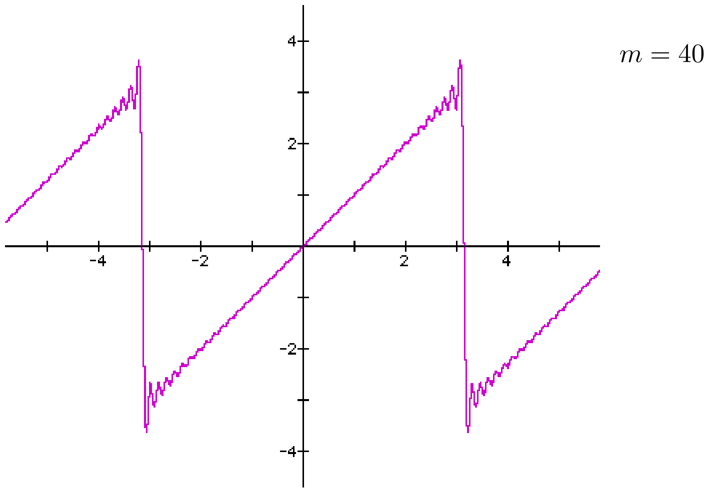
$m = 10$



$m = 20$

Observe that $f(x) \to \pi$ as $x \to \pi - 0$ and $f(x) \to -\pi$ as $x \to -\pi + 0$. Observe also that $S_m(\pm\pi) = 0$ for every $m \in \mathbb{N}$, and so the Fourier series has value 0 at these points. Let us look at two more graph, representing the partial sums $S_m(x)$ for $m = 30, 40$:





We may still expect $S_m(x) \to x$ for every $x \in (-\pi, \pi)$. However, it is also clear that there is some erratic behaviour of the sequence of partial sums near the points of discontinuity $x = \pm\pi$ of the function f.

EXAMPLE 1.2.4. Consider the function $f : [-\pi, \pi] \to \mathbb{C}$, given for every $x \in [-\pi, \pi]$ by

$$f(x) = \operatorname{sgn}(x) = \begin{cases} +1, & \text{if } 0 < x \leqslant \pi, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } -\pi \leqslant x < 0. \end{cases}$$

We shall show in Example 3.1.3 that this function has Fourier series

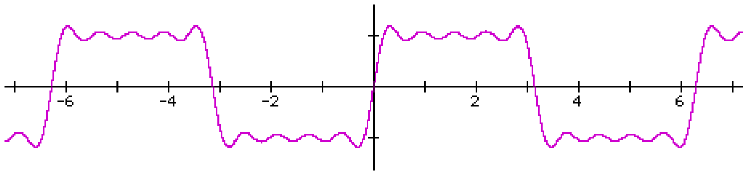
$$\operatorname{sgn}(x) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{\pi n} \sin nx,$$

with partial sums

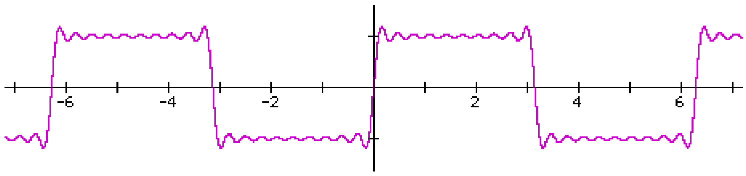
$$S_m(x) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{m} \frac{4}{\pi n} \sin nx.$$

Again the Fourier series and all the partial sums are periodic functions with period 2π. As in the last example, by altering our definition of f at one or both of the points $x = \pm\pi$, we may extend f to a periodic function with period 2π on the real line $\mathbb{R}$ by writing $f(x + 2\pi) = f(x)$ for every $x \in \mathbb{R}$.

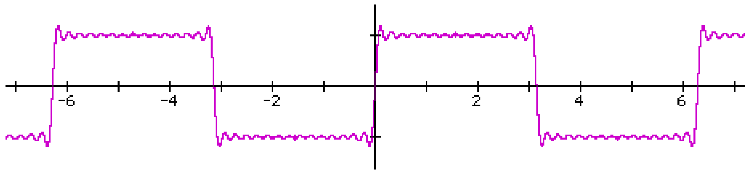
The following graphs represent the partial sums $S_m(x)$ for $m = 9, 19, 29, 39$:



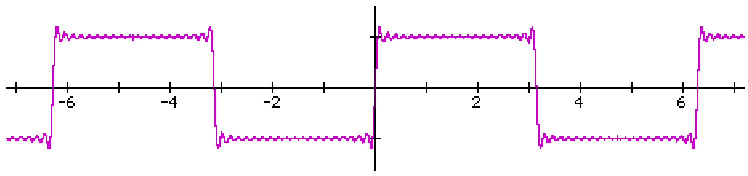
$m = 9$



$m = 19$



$m = 29$



$m = 39$

Note that the erratic behaviour of the partial sums near the points of discontinuity $x = \pm\pi$ of the function f is better illustrated than in the last example. We may still expect $S_m(x) \to \mathrm{sgn}(x)$ for every $x \in (-\pi, \pi)$. However, these last two graphs suggest that the erratic behaviour of the sequence of partial sums near the points of discontinuity $x = \pm\pi$ of the function f may not go away, but simply shifts closer to the points of discontinuity $x = \pm\pi$. This erratic behaviour is known as the Gibbs phenomenon.